Randomized Isometric Linear Dispersion Space-Time Block Coding for the DF Relay Channel

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***Abstract*—This paper presents a randomized linear-dispersion space-time block code for decode-and-forward synchronous relays. The coding matrices are obtained as a set of columns (or rows) of randomly generated Haar-distributed unitary matrices. With respect to independent and identically distributed (i.i.d.)-generated codes, this particular isometric structure reduces the intersymbol interference generated within each relay. The gain over i.i.d. codes in terms of spectral efficiency is analyzed for both the LMMSE and the ML receivers under the assumption of frequency-flat quasi-static fading. In this setting, the spectral efficiency is a random quantity, since it depends on the random coding matrices. However, it is proven that the spectral efficiency converges in probability to a deterministic quantity when the dimensions of the matrices tend to infinity while keeping constant their ratio, i.e., the coding rate α. Consequently, when the random coding matrices are large enough, the presented system behaves as a deterministic one. This result is achieved by means of the rectangular R-transform, a powerful tool of free probability theory which allows determining the distribution of the singular values of a sum of rectangular matrices.**

***Index Terms*—DF relay channel, free probability, randomized isometric linear-dispersion space-time block coding (LD-STBC), random matrix theory, rectangular free additive convolution.**

I. Introduction

RELAY communications have raised a lot of interest in the last years as a potential means of introducing space-diversity techniques [2]–[5] in systems where the limited dimensions of portable terminals prevent them from having multiple co-located antennas (see [6]–[9] among others). The main idea is that idle terminals overhear other users' communications and, thus, can act as relays, forwarding the information they receive. In other words, a virtual array is built from multiple single-antenna terminals.

To reduce power consumption and signaling among terminals, relays should be low-complexity devices. It is therefore reasonable to assume they do not have any channel state information, especially in their transmitting phase. In these circumstances, previous experience in multiple-input-multiple-output (MIMO) systems (e.g., [2]) leads us to believe that space-time coding (STC) is one of the best options to achieve full spatial diversity. References [10]–[14] are just few examples on that direction.

Classical STC's, however, are not very suitable for most relay networks. On the one hand, the design of a code is strongly related to the number of transmitters, and its complexity increases with the latter. On the other hand, modern mobile communications networks are very dynamic, with users continuously dropping in and out of the system and where the total number of terminals may possibly be quite large. It would hence be advisable to implement a code which is flexible and easy to design, even for a large and time-varying number of users.

A. Previous Work

This need for flexibility in wireless relay networks has long been known. In [11], Laneman and Wornell suggest employing space-time codes from orthogonal designs as a possible solution. These space-time block codes, originally proposed by Tarokh *et al.* [4], are designed for a given number L of transmitters but maintain their orthogonality properties when some of the antennas are shut down. This implies that the maximum number of relays in the system must be known *a priori*. Moreover, for more than four transmitters, the coding rate is only 1/2, thus limiting the spectral efficiency.

The solution proposed in [13] is based on linear-dispersion space-time block coding (LD-STBC): each relay is assigned a specific unitary matrix which produces a linear transformation of the vector of source symbols. The system is quite flexible, since no particular relation is assumed among the different coding matrices: when a new terminal joins the network, a new matrix is generated without modifying the existent ones. In this paper, however, no direct link between the transmission source and its destination is considered. Furthermore, the choice of unitary matrices constrains the coding rate (defined here as the number of columns divided by the number of rows of these matrices) to one. As shown in [14], this is not always the best choice for half-duplex relays: it may be enough for the relays to send a compressed version of the message (i.e., coding rate larger than one), since they only complement the information received directly from the source. This is especially true for orthogonal relaying protocols, that is when the source remains silent during the relaying phase.

A completely different approach appears in [12]. From the source message, each relay generates a new vector of symbols

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by doing a random linear combination of the columns of a matrix codeword, which is obtained from a common (within the relay set) deterministic space-time mapping. It turns out that the system performance is limited by the minimum between the number of relays and the number of virtual transmitters of the underlying deterministic STC.

B. The Proposed Scheme

The coding scheme presented in [14] is an LD-STBC where the coding matrices are filled with entries that are drawn from independent and identically distributed (i.i.d.) random variables. In this paper, we try to improve the spectral efficiency by introducing codes with more structure, while still randomly generated. More specifically, the columns (or the rows, respectively) of the matrices are constrained to be orthogonal: the aim is to cancel (or to reduce, respectively) interference generated within the relays. For coding rate α (ratio between the number of columns K and the number of rows N of the linear-dispersion matrices) equal to one, the linear-dispersion matrices are unitary and the system is similar to the one in [13]. However, simulation results show that this trivial choice is not always the best one.

Using a similar approach as in [14], we analyze the asymptotic performance of the system assuming coding matrices with infinitely large dimensions, but constant coding rate α, $0 < \alpha < +\infty$. Indeed, in this asymptotic regime, the spectral efficiency converges to a deterministic value which is an excellent approximation of the finite reality, even for not-so-large dispersion matrices. Contrary to the i.i.d. case, however, classical random matrix theory results on the convergence of the eigenvalues of infinite-dimensional matrices are not enough to characterize the asymptotic behavior of the system. New tools are borrowed from free probability in order to deal with the present problem.

The paper is structured as follows. In the next section, the signal model is introduced and all the assumptions are presented. Then, general expressions of the spectral efficiency are derived for both the linear minimum-mean-square-error (LMMSE) and the maximum-likelihood (ML) receivers, together with their asymptotic equivalents for large coding matrices. Next, Section IV analyzes two special cases with closed-form solution and introduces a low-complexity approximation for the general case. Section V presents the rectangular R-transform introduced by Benaych-Georges in [15] and shows how to apply this free-probability tool to compute the asymptotic spectral efficiencies of the two receivers considered here. A numerical assessment of the results is given in Section VI, while Section VII characterizes the low-power regime. Finally, Section VIII concludes the paper.

II. System Description

This section provides a more thorough description of the system under consideration. As usual, italic, bold lower-case and bold upper-case letters denote, respectively, scalars, vectors and matrices. The superscipts $*$, ^{T} and ^{H} stand for, respectively, complex conjugate, transpose and Hermitian transpose. $\mathbb{E}[\cdot]$ is the statistical expected value operator. Given any integer number n, $\mathbf{I}_n$ is the $n \times n$ identity matrix.

A. Signal Model

We consider a classical multiple-relay system with half-duplex synchronous relays over frequency-flat quasi-static fading channels [10], [11]. Communications are split into two phases: the source broadcasts its message in the first phase and remains silent in the second one, which is used by the relays to forward the information they have just received. In this relaying phase, space diversity is achieved by means of a spatially distributed LD-STBC, as in [13], [14]. Here, however, relays adopt a decode-and-forward (DF) strategy, whereby only the terminals that can correctly decode the source message in the first phase participate in the second one. Note that the proposed coding scheme may also be applied to amplify-and-forward (AF) relays (all terminals forward the whole signal received in the first phase, including noise). However, the analysis will be much more complex, due to the forwarded noise.

Let $\mathbf{s} = [s_1 \ \cdots \ s_K]^T$ denote the vector containing the message from the source. The symbols s_k, $k = 1, \ldots, K$, are assumed i.i.d., with zero mean ($\mathbb{E}[s_k] = 0$) and variance P_s ($\mathbb{E}[|s_k|^2] = P_s$). Let $\mathcal{L}$ denote the set of all relays and $\mathcal{L}'$ the decoding subset, i.e., the set of relays that are able to decode the source message, have a perfect copy of $\mathbf{s}$ at the end of the first transmission phase and, thus, participate in the relaying phase. Let $L = |\mathcal{L}'|$ be the cardinality of $\mathcal{L}'$ and, without loss of generality, $\mathcal{L}' = \{1, \ldots, L\}$. Now, denoting by h_s the direct source-destination link, the destination receives the vector $h_s\mathbf{s}$ during the first phase. Next, in the second phase, the lth relay in $\mathcal{L}'$ transmits $g_l\mathbf{C}_l\mathbf{s}$, where $\mathbf{C}_l$ is the $N \times K$ encoding matrix described below. The complex gain g_l can be set to fulfill some power constraint. Denoting by h_{dl} the downlink channel coefficient between the lth relay and the destination, the received signal can be written as

$$\mathbf{d} = \begin{bmatrix} h_s\mathbf{I}_K \\ \tilde{\mathbf{\Psi}}\tilde{\mathbf{C}} \end{bmatrix} \mathbf{s} + \mathbf{n} \tag{1}$$

where both transmission phases have been included in the formulation. In the previous equation, $\mathbf{n} \sim \mathcal{CN}(\mathbf{0}, \sigma_d^2\mathbf{I}_{K+N})$ represents additive white Gaussian noise and the block matrices

$$\tilde{\mathbf{\Psi}} = [\, g_1 h_{d1}\mathbf{I}_N \quad \cdots \quad g_L h_{dL}\mathbf{I}_N \,] \qquad \tilde{\mathbf{C}} = [\, \mathbf{C}_1^T \quad \cdots \quad \mathbf{C}_L^T \,]^T$$

have been introduced to simplify notation (i.e., $\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}} = \sum_{l=1}^{L} g_l h_{dl}\mathbf{C}_l$).

In the following sections, the spectral efficiency $I^{(\mathcal{L}')}$ (conditioned on the decoding subset $\mathcal{L}'$) is computed both for the LMMSE receiver and the optimum ML receiver assuming that the channels and the codes are known at the destination (observe that the quasi-static fading assumption implies that the channel coefficients are constant over the transmission of the whole source message $\mathbf{s}$). A direct application of this result is the outage-probability analysis for a target transmission rate R. Indeed, according to the total probability theorem

$$P_{\text{out}}(R) = \sum_{\mathcal{L}' \subseteq \mathcal{L}} \Pr\left[I^{(\mathcal{L}')} < R\right] \Pr[\mathcal{L}' \text{ is the decoding set}] \tag{2}$$

where the sum is over all possible subsets of $\mathcal{L}$ and probabilities follow from channel distribution. However, this paper focuses on the spectral efficiency and its properties, leaving the nontrivial outage analysis for further contributions (see also Section VI).

B. The Coding Matrices

The introduced coding scheme is similar to the one presented in [16] or in [14] for AF relays. As aforementioned in the Introduction, however, we propose here a different model for the coding matrices $\mathbf{C}_l$.

Observe that the vector corresponding to the signal generated by the lth relay is given by $\mathbf{C}_l \mathbf{s} = \sum_{k=1}^{L} \mathbf{c}_{l,k} s_k$, where $\mathbf{c}_{l,k}$ is the kth column of $\mathbf{C}_l$. Thus, a straightforward analogy with DS/CDMA systems [17] suggests that the intersymbol interference at the receiver will be reduced when the columns of $\mathbf{C}_l$ are orthogonal. Note that interference will not completely vanish since orthogonality is required only within each individual relay. Extending the constraint across all the relays would imply a very significant loss of flexibility, since all matrices would have to be jointly designed to be mutually orthogonal. Here, we are more interested in a dynamic system where the number of active terminals can vary without significantly jeopardizing the global coding scheme. Furthermore, global orthogonality is equivalent to TDMA, which is shown to be outperformed by STC in [11].

As a result, we model the $N \times K$ matrices $\{\mathbf{C}_l : l = 1, \ldots, L\}$ as mutually independent random matrices with orthogonal columns. More specifically, each coding matrix is constructed by selecting K different columns of a $N \times N$ Haar-distributed unitary matrix, i.e., a random unitary matrix whose distribution is invariant by left- or right-multiplication by a constant unitary matrix (we say that it is bi-unitarily invariant). We will refer to this model as Haar codes or, equivalently, as random isometric codes.

A matrix with orthogonal columns must be such that $K \leq N$, i.e., $\alpha = K/N \leq 1$. For completeness, the analysis is extended to the case $\alpha > 1$ by considering coding matrices with orthogonal rows (N rows of a $K \times K$ Haar-distributed unitary matrix). A scaling factor $\sqrt{\alpha}$, such that $\mathbf{C}_l \mathbf{C}_l^H = \alpha \mathbf{I}_N$, must be included to guarantee that the same power constraint as in the i.i.d. case is satisfied, which leads to a fair comparison of the results for the two different choices of the dispersion matrices. Although not as intuitive as the case $\alpha \leq 1$, fat linear dispersion matrices still improve spectral efficiency with respect to i.i.d. codes of, e.g., [14], [16]. The reason is that, whatever the value of α, isometric codes result in an equivalent channel matrix that is closer to an identity matrix (the difference between maximum and minimum singular values is lower).

To conclude this introduction, let us mention that Haar codes have been extensively studied in the literature (see, for instance, [18]–[20]). In particular, [18] explains how to generate Haar-distributed matrices from a random matrix with i.i.d. Gaussian entries having zero mean and unitary variance.

III. Main Results

In this section, the spectral efficiency of the presented system is computed assuming that the destination has access to the code matrix and the channel coefficient of each active relay. Two classical receivers are considered, namely the LMMSE receiver and the ML receiver.

A. Spectral Efficiency

1) The LMMSE Receiver: It is well known that the LMMSE filter is the best linear receiver in terms of signal-to-interference-plus-noise ratio (SINR) (see, e.g., [21]). Since each symbol s_k is estimated independently of the others, let us focus on the first one, s_1, without loss of generality, and rewrite (1) as $\mathbf{d} = \mathbf{a} s_1 + \mathbf{n}_E$, where

$$\mathbf{a} = \begin{bmatrix} h_s \\ \mathbf{0} \\ \tilde{\mathbf{\Psi}} \mathbf{c}_1 \end{bmatrix} \quad \text{and} \quad \mathbf{n}_E = \begin{bmatrix} \mathbf{0} \\ h_s \mathbf{I}_{K-1} \\ \tilde{\mathbf{\Psi}} \mathbf{D} \end{bmatrix} \begin{bmatrix} s_2 \\ \vdots \\ s_K \end{bmatrix} + \mathbf{n}$$

are the effective channel seen by the symbol s_1 and the equivalent interference-plus-noise vector, respectively. In these definitions, we have introduced the vector $\mathbf{c}_1$ and the matrix $\mathbf{D}$ such that $\tilde{\mathbf{C}} = [\mathbf{c}_1 \mathbf{D}]$. Now, the LMMSE filter coefficients and the corresponding output SINR can be written, respectively, as

$$\mathbf{w} = \frac{P_s}{1 + P_s \mathbf{a}^H \mathbf{R}_E^{-1} \mathbf{a}} \mathbf{R}_E^{-1} \mathbf{a} \quad \mathrm{SINR}_1 = P_s \mathbf{a}^H \mathbf{R}_E^{-1} \mathbf{a}$$

where $\mathbf{R}_E = \mathbb{E}[\mathbf{n}_E \mathbf{n}_E^H]$. After some algebra, it is straightforward to show that the SINR of the considered system can be expressed as

$$\mathrm{SINR}_1 = \frac{P_s}{\sigma_d^2} |h_s|^2 + \frac{P_s}{\sigma_d^2} \mathbf{c}_1^H \tilde{\mathbf{\Psi}}^H \times \left(\frac{P_s/\sigma_d^2}{1 + P_s |h_s|^2/\sigma_d^2} \tilde{\mathbf{\Psi}} \mathbf{D} \mathbf{D}^H \tilde{\mathbf{\Psi}}^H + \mathbf{I}_N \right)^{-1} \tilde{\mathbf{\Psi}} \mathbf{c}_1 \quad (3)$$

for symbol s_1 and analogously for the other symbols.

Considering the contribution of all the symbols, the spectral efficiency can be computed by means of the Shannon's formula as

$$I_{\mathrm{LMMSE}} = \frac{1}{K+N} \sum_{k=1}^{K} \ln(1 + \mathrm{SINR}_k) \quad (4)$$

in nats per degree of freedom. The factor $\frac{1}{K+N}$ takes into account the fact that a total of $(K+N)$ channel accesses are employed to transmit only K information symbols.

2) The ML Receiver: For a system with colored interference like the one presented in this paper, linear filters are suboptimal receivers. To extract all the information contained in the received signal $\mathbf{d}$, the ML receiver is needed. Assuming independent Gaussian coding at the source, the spectral efficiency of the ML receiver in our scenario is known to be [5]

$$\begin{aligned} I_{\mathrm{ML}} &= \frac{1}{K+N} \ln \det \Bigg(\mathbf{I}_{K+N} + \frac{P_s}{\sigma_d^2} \begin{bmatrix} h_s \mathbf{I}_K \\ \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}} \end{bmatrix} \begin{bmatrix} h_s^* \mathbf{I}_K & \tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \end{bmatrix} \Bigg) \\ &= \frac{\alpha}{1+\alpha} \ln \left(1 + \frac{P_s}{\sigma_d^2} |h_s|^2 \right) \\ &\quad + \frac{1}{K+N} \ln \det \left(\mathbf{I}_K + \frac{\frac{P_s}{\sigma_d^2}}{1 + \frac{P_s}{\sigma_d^2} |h_s|^2} \tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}} \right) \end{aligned} \quad (5)$$

in nats per degree of freedom.

B. Asymptotic Results

Observe that both the spectral efficiencies in (4) and (5) are random quantities, since they intrinsically depend on the randomly generated coding matrices $\{\mathbf{C}_l\}$. In other words, for each realization of the code, the system performs differently. However, as proven in Section V, both I_{LMMSE} and I_{ML} quickly converge to deterministic quantities when the dimensions K and N of the linear-dispersion matrices grow indefinitely while keeping constant their ratio, that is the coding rate α. Before giving more details, we need to introduce some useful quantities.

The asymptotic behavior of I_{LMMSE} and I_{ML} is dictated by the asymptotic distribution of the eigenvalues of $\tilde{\mathbf{C}}^H\tilde{\mathbf{\Psi}}^H\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}$, which is studied next. More specifically, we will now characterize the asymptotic eigenvalue distribution in terms of its moments. Then, we will show that these moments are sufficient to compute the asymptotic spectral efficiencies for large K and N.

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_K$ be the K real nonnegative eigenvalues of the $K \times K$ interference matrix $\tilde{\mathbf{C}}^H\tilde{\mathbf{\Psi}}^H\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}$. We define the empirical distribution of the eigenvalues as follows[1]:

$$\nu_N^2 = \frac{1}{K}\sum_{k=1}^{K}\delta_{\lambda_k} \tag{6}$$

where δ_λ is the Dirac distribution (mass point) at λ. Since the eigenvalues are generally random, their empirical distribution is also random. However, the following theorem states that the distribution ν_N^2 converges when $K = \alpha N \to +\infty$, i.e., K and N grow indefinitely while their ratio tends[2] to α.

Theorem 3.1: Let $K, N \to +\infty$ and $K/N \to \alpha \in (0, 1]$. Then, the empirical eigenvalue distribution ν_N^2 of the interference matrix $\tilde{\mathbf{C}}^H\tilde{\mathbf{\Psi}}^H\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}$ converges weakly in probability to a probability measure ν^2 (and we write $\nu_N^2 \xrightarrow{\mathrm{P}} \nu^2$), meaning that

$$\lim_{K=\alpha N\to+\infty}\int f(t)\nu_N^2(\mathrm{d}t) = \int f(t)\nu^2(\mathrm{d}t) \tag{7}$$

in probability for any continuous and bounded function $f(\cdot)$.

Moreover:

- the support of ν^2 is compact and included in $[0, \max_l\{|g_l h_{dl}|^2\}]$;
- naming $\{m_i = \int t^i \nu^2(\mathrm{d}t);\ i = 1, 2, \ldots\}$ the moments of ν^2, the moment generating series $M_{\nu^2}(z) = \sum_{i=1}^{+\infty} m_i z^i$ of ν^2 is the unique formal power series that satisfies the fixed point equation
$$M_{\nu^2}(z) = C_\nu[zT(M_{\nu^2}(z))]$$
where $C_\nu(z) = \sum_{i=1}^{+\infty} c_i z^i$ is the formal power series with ith coefficient
$$c_i = \frac{(2\alpha)^{i-1}}{i!}\left(\prod_{k=0}^{i-1}(1-2k)\right)\sum_{l=1}^{L}|g_l h_{dl}|^{2i}$$
and where $T(z) = (\alpha z + 1)(z + 1)$.

[1]We use the square in ν_N^2 in order to emphasize that it is a law on the eigenvalues, and not on the singular values.

[2]With some abuse of notation, we denote by α both the finite coding rate $\alpha = K/N$ for finite K and N and the asymptotic coding rate $\frac{K}{N} \to \alpha$, when both K and N tend to infinity. For each particular instance, the context clarifies the meaning.

Proof: See Section V. ■

Note that the sequence of moments $\{m_i : i = 1, 2, \ldots\}$ is sufficient to characterize and univocally identify the distribution ν^2 since the latter has a compact support. Furthermore, the power series $M_{\nu^2}(z) = \sum_{i=1}^{+\infty} m_i z^i$ can be seen as a series expansion near $z = 0$ of the Moment Generating Function[3] (MGF) of ν^2, defined as

$$M_{\nu^2}(z) = \int \frac{zt}{1-zt}\nu^2(\mathrm{d}t) \tag{8}$$

and analytic on $\mathbb{C} \setminus \mathbb{R}_+$.

Note. The interference matrix $\tilde{\mathbf{C}}^H\tilde{\mathbf{\Psi}}^H\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}$ is full rank only if $K \leq N$ ($\alpha \leq 1$). Conversely, when $K > N$ the matrix has $K - N$ null eigenvalues and N positive eigenvalues that are equal to those of $\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}\tilde{\mathbf{C}}^H\tilde{\mathbf{\Psi}}^H$. Then, when $\alpha > 1$, let μ^2 be the eigenvalue distribution of the full rank matrix $\tilde{\mathbf{\Psi}}\mathbf{W}\mathbf{W}^H\tilde{\mathbf{\Psi}}^H$, where we have defined $\mathbf{W} = \frac{1}{\sqrt{\alpha}}\tilde{\mathbf{C}}$ (recall that $\mathbf{C}_l\mathbf{C}_l^H = \alpha\mathbf{I}_N$ when $\alpha > 1$). Denoting by δ_0 the Dirac delta distribution, the distribution ν^2 is related to μ^2 by the following identity:

$$\nu^2(\mathrm{d}t) = \frac{\alpha-1}{\alpha}\delta_0(\mathrm{d}t) + \frac{1}{\alpha}\mu^2\left(\frac{\mathrm{d}t}{\alpha}\right)$$

or, equivalently

$$M_{\nu^2}(z) = \frac{1}{\alpha}M_{\mu^2}(\alpha z).$$

Since it is totally equivalent to characterize the distribution ν^2 when $\alpha \leq 1$ or the distribution μ^2 when $\alpha > 1$, we concentrate on the case $\alpha \leq 1$ in what follows.

Knowing that the asymptotic eigenvalue distribution exists and is unique, we have all the necessary tools to study the asymptotic deterministic spectral efficiencies (see also [1]).

Theorem 3.2: Consider the relay channel described in Section II-A, with the isometric linear-dispersion matrices defined in Section II-B. When the matrix dimensions K and N grow without bound but with constant coding rate $\alpha = K/N$, the LMMSE spectral efficiency (4) converges in probability to the deterministic quantity

$$I_{\mathrm{LMMSE}}^{\mathrm{Haar}} = \frac{\alpha}{1+\alpha}\ln(1 + \mathrm{SINR}^{\mathrm{Haar}}) \tag{9}$$

where

$$\mathrm{SINR}^{\mathrm{Haar}} = \frac{P_s}{\sigma_d^2}|h_s|^2 + \frac{P_s}{\sigma_d^2}\frac{\eta^{\mathrm{Haar}}}{1-\chi\eta^{\mathrm{Haar}}} \tag{10}$$

$$\chi = \frac{P_s/\sigma_d^2}{1 + P_s|h_s|^2/\sigma_d^2} \quad \text{and}$$

$$\eta^{\mathrm{Haar}} = -\frac{1}{\chi}M_{\nu^2}(-\chi).$$

Similarly, the deterministic limit of the ML spectral efficiency (5) is

$$I_{\mathrm{ML}}^{\mathrm{Haar}} = \frac{\alpha}{1+\alpha}\ln\left(1 + \frac{P_s}{\sigma_d^2}|h_s|^2\right) + \frac{\alpha}{1+\alpha}\int_{-\chi}^{0}\frac{M_{\nu^2}(u)}{u}\mathrm{d}u. \tag{11}$$

[3]Note that this is different from the classical moment generating function, usually defined as $\mathbb{E}_X[e^{zX}]$.

Proof: See Section V. ■

As explained above, the main motivation behind Haar-distributed random coding is the interference reduction with respect to i.i.d. coding introduced in, e.g., [14]. Thus, for the sake of comparison, we report here the asymptotic spectral efficiencies of the i.i.d. case.

Theorem 3.3: Consider the relay channel described in Section II-A, and assume that the linear-dispersion matrices $\{\mathbf{C}_l\}$ are filled with i.i.d. random variables with zero mean and variance $1/N$. The spectral efficiency for the LMMSE receiver and the ML receiver are given by (4) and (5), respectively. Then, when $K, N \to +\infty$ while $K/N \to \alpha$, one has

$$I_{\mathrm{LMMSE}} \overset{\text{a.s.}}{\to} I_{\mathrm{LMMSE}}^{\mathrm{iid}} = \frac{\alpha}{1+\alpha} \ln\left(1 + \frac{P_s}{\sigma_d^2}|h_s|^2 + \frac{P_s \sum_{l=1}^{L} |g_l h_{dl}|^2}{\beta \sigma_d^2}\right), \tag{12a}$$

$$I_{\mathrm{ML}} \overset{\text{a.s.}}{\to} I_{\mathrm{ML}}^{\mathrm{iid}} = \frac{\alpha}{1+\alpha} \ln\left(1 + \frac{P_s}{\sigma_d^2}|h_s|^2 + \frac{P_s \sum_{l=1}^{L} |g_l h_{dl}|^2}{\beta \sigma_d^2}\right) + \frac{1}{1+\alpha}\left(\ln\beta + \frac{1}{\beta} - 1\right) \tag{12b}$$

where β is the positive solution to

$$\beta = 1 + \alpha\beta \frac{\chi \sum_{l=1}^{L} |g_l h_{dl}|^2}{\beta + \chi \sum_{l=1}^{L} |g_l h_{dl}|^2}$$

which is given by (13) shown at the bottom of the page.

Proof: These results can be proven following the same guidelines given in [14] for AF relays (see also [16] and [22]).■

In summary, large Haar-distributed (and i.i.d.) random linear-dispersion matrices asymptotically behave as deterministic systems. Since convergence is fast, the limiting spectral efficiencies are excellent approximations of the finite reality for practical values of K and N, as evidenced from the numerical simulations in Section VI.

Observe that the limits in (12) hold almost surely and not only in probability as those derived above for the isometric coding scheme. It seems only natural to conjecture that these convergence results also hold in the almost-sure sense. However, the mathematical background used here has only been able to establish convergence in probability; besides, the difference between the two convergence modes has no real importance in practical aspects.

Before delving into the mathematical details of the proofs of Theorems 3.1 and 3.2 (which are not very insightful from the engineering point of view), let us get some more understanding about the asymptotic eigenvalue distribution and the consequent large K, N spectral efficiencies.

IV. MGF Derivation and Special Cases

Theorem 3.2 expresses the asymptotic spectral efficiencies in terms of $M_{\nu^2}(z)$, the MGF of the asymptotic eigenvalue distribution ν^2 of the interference matrix $\tilde{\mathbf{C}}^H\tilde{\mathbf{\Psi}}^H\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}$. Even though Theorem 3.1 univocally defines $M_{\nu^2}(z)$, it does not describe an operative algorithm to evaluate it as a function of z at any $z \in [-\chi, 0)$. Indeed, $M_{\nu^2}(z)$ is defined in Theorem 3.1 as a formal power series, which may not accept a closed-form analytical representation for all $z \in [-\chi, 0)$. The following lemma provides us with a more practical solution.

Lemma 4.1: For any $z \in \mathbb{R}_-$, the MGF $M_{\nu^2}(z)$ satisfies the system of equations

$$M_{\nu^2}(z) = \sum_{l=1}^{L} M_l(z),$$

$$\begin{bmatrix} M_1(z)(1+\alpha M_1(z)) \\ \vdots \\ M_L(z)(1+\alpha M_L(z)) \end{bmatrix} = z(1+\alpha M_{\nu^2}(z)) \times (1 + M_{\nu^2}(z)) \begin{bmatrix} |g_1 h_{d1}|^2 \\ \vdots \\ |g_L h_{dL}|^2 \end{bmatrix} \tag{14}$$

together with the constraints

$$-1 < M_{\nu^2}(z) \le 0 \quad -1 < M_l(z) \le 0,\ l = 1, \ldots, L.$$

Proof: See Appendix A.A. ■

The system in (14) is not linear and, hence, its solution may not be trivial. In what follows, two special cases that accept closed-form solutions are presented, namely (i) when there are only $L = 2$ relays in the system and (ii) when all the relay-destination channels are equal.

A. Some Special Cases

We first consider the two-relay case. As shown in Section VI, it is in this case that isometric codes present the highest gain over i.i.d. codes.

Proposition 4.1: Consider the relay channel presented in Section II-A. Assume that Haar-distributed random LD-STBC is employed and that the number of relays is $L = 2$. Then, the MGF of the asymptotic eigenvalue distribution of the interference matrix $\tilde{\mathbf{C}}^H\tilde{\mathbf{\Psi}}^H\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}$ is given by (15) at the bottom of the next page, where we have written $\gamma^{(+)} = |g_1 h_{d1}|^2 + |g_2 h_{d2}|^2$ and $\gamma^{(-)} = |g_1 h_{d1}|^2 - |g_2 h_{d2}|^2$.

Proof: See Appendix A.B. ■

As aforementioned in the Introduction, it is clear from (15) that the asymptotic spectral efficiency of the considered system (which is expressed in terms of $M_{\nu^2}(z)$, see (9) and (11)) depends neither on the specific instance of the code nor directly

$$\beta = \frac{1 - (1-\alpha)\chi \sum_{l=1}^{L} |g_l h_{dl}|^2 + \sqrt{\left(1 - (1-\alpha)\chi \sum_{l=1}^{L} |g_l h_{dl}|^2\right)^2 + 4\chi \sum_{l=1}^{L} |g_l h_{dl}|^2}}{2}. \tag{13}$$

on its size, but only depends on the coding rate α and on the relay-destination channel gains.

A simpler expression can be found when assuming that all the effective downlink channels are equal. This is clearly a purely didactic case study, since there exists a null probability that all relay-receiver channels are equal (see also [1]). However, it allows us to get some insight into the isometric coding scheme when comparing with the i.i.d. case. Indeed, the comparison readily extends to the general case, as it will be clear in the following sections.

Proposition 4.2: Consider the relay channel presented in Section II-A. Assume that Haar-distributed random LD-STBC is employed and that $|g_l h_{dl}|^2 = 1$, $l = 1, \ldots, L$. Then, the MGF of the asymptotic eigenvalue distribution of the interference matrix $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{C}} \tilde{\mathbf{\Psi}}$ is given by

$$M_{\nu^2}(z) = \frac{L}{2\alpha(1-L^2 z)}[L(\alpha+1)z - 1 + \sqrt{(L(1-\alpha)z+1)^2 - 4(L-\alpha)z}]. \quad (16)$$

Proof: See Appendix A.C. ■

As a remark, it is straightforward to verify that (16) and (15) represent the same function when setting $L = 2$ and $|g_1 h_{d1}|^2 = |g_2 h_{d2}|^2 = 1$ (i.e., $\gamma^{(-)} = 0$ and $\gamma^{(+)} = 2$).

In the general case, the solution of the nonlinear system in (14) can give rise to equations whose order increases fast with the number of relays. To avoid having to deal with such complicated equations, we propose next a low-complexity procedure to approximate $M_{\nu^2}(z)$.

B. A Moment-Based Approximation

We have already mentioned (see Theorem 3.1) that the distribution ν^2 of the eigenvalues of $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$ has a compact real support, contained in $[0, \max_l\{|g_l h_{dl}|^2\}]$, and, hence, is univocally determined by the sequence of its moments $\{m_i, i = 1, 2, \ldots\}$.

The basic idea behind the approximation proposed hereafter is to replace the measure ν^2 by a new discrete measure of the form $\bar{\nu}_n^2 = \sum_{k=1}^n \gamma_{k,n} \delta_{\lambda_{k,n}}$, such that their first $2n-1$ moments are respectively equal, where $n \in \mathbb{N}_+$ will depend on the allowed complexity (see also [1]). For every n, the points $\{\lambda_{k,n}\}_{k=1}^n$ should fall in the support of ν^2 and, obviously, $\gamma_{k,n} > 0, \forall k \in \{1, \ldots, n\}$, with $\sum_{k=1}^n \gamma_{k,n} = 1$. The motivation behind this choice is that well-known results on the moment problem [23]–[25] tell us that the point-wise convergence $\bar{M}_n(z) \to M_{\nu^2}(z)$ is exponential in n, where we denote by

$$\bar{M}_n(z) = z \sum_{k=1}^n \frac{\gamma_{k,n}\lambda_{k,n}}{1 - z\lambda_{k,n}}$$

the MGF of $\bar{\nu}_n^2$. Thus, we propose the following approximations:

$$\eta^{\text{Haar}} \approx \sum_{k=1}^n \frac{\gamma_{k,n}\lambda_{k,n}}{1+\chi\lambda_{k,n}},$$

$$I_{\text{ML}}^{\text{Haar}} \approx \frac{\alpha}{1+\alpha} \ln\left(1 + \frac{P_s}{\sigma_d^2}|h_s|^2\right) + \frac{\alpha}{1+\alpha}\sum_{k=1}^n \gamma_{k,n} \ln(1+\chi\lambda_{k,n}).$$

Observe that, according to the last equation, the proposed approximation is equivalent to splitting the transmission over n parallel channels, the kth one having channel gain $\lambda_{k,n}$ and carrying a fraction $\gamma_{k,n}$ of the total information.

1) The Gauss-Jacobi Mechanical Quadrature: It remains to explain how to compute the coefficients $\gamma_{k,n}$ and $\lambda_{k,n}$. The problem of approximating $M_{\nu^2}(z)$ by $\bar{M}_n(z)$ is known in the literature as the Gauss-Jacobi mechanical quadrature and makes use of the theory of orthogonal polynomials [23], [24]. We summarize hereafter its main points.

For the probability measure ν^2 with moments $m_i = \int t^i \nu^2(\mathrm{d}t)$, we define the scalar product

$$\langle f, g \rangle = \int f(\lambda) g(\lambda) \nu^2(\mathrm{d}\lambda)$$

on the space[4] $L^2(\nu^2)$. Then, the Gram-Schmidt orthogonalization procedure can be applied to the sequence of polynomials $\{\lambda^n : n = 1, 2, \ldots\}$, defined as nonnegative powers of λ. As a result, we get a sequence $\{p_n(\lambda)\}_{n \geq 0}$ such as

- the polynomial $p_n(\lambda)$ has degree n and positive leading coefficient;
- the polynomials are orthonormal, i.e., $\langle p_n, p_q \rangle = 1$ if and only if $n = q$ and zero otherwise.

Equivalently, the polynomials $p_n(\lambda)$ can be computed recursively, thanks to the following result.

Proposition 4.3 (The Three Terms Recursion Relation [24]): The family of polynomials $\{p_k\}$ satisfies

$$\lambda p_k(\lambda) = b_{k-1} p_{k-1}(\lambda) + a_k p_k(\lambda) + b_k p_{k+1}(\lambda)$$

[4]Recall that, given a measure space (S, Σ, μ), the space $L^2(\mu)$ is, roughly speaking, the vector space of all functions $f(\cdot)$ such that $\int_S |f(t)|^2 \mu(\mathrm{d}t) < +\infty$.

$$M_{\nu^2}(z) = -\frac{\left[1+\alpha\left(z\gamma^{(-)}\right)^2 - z(1+\alpha)\gamma^{(+)}\right]}{\alpha\left[1+\left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)}\right]}\left[1 - \sqrt{1 - \frac{\alpha\left[\alpha\left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)}\right]\left[1+\left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)}\right]}{\left[1+\alpha\left(z\gamma^{(-)}\right)^2 - z(1+\alpha)\gamma^{(+)}\right]^2}}\right] \quad (15)$$

where the coefficients a_k and b_k, defined by $a_k = \langle \lambda p_k(\lambda), p_k(\lambda)\rangle$ and $b_k = \langle \lambda p_k(\lambda), p_{k+1}(\lambda)\rangle$, are positive. The recurrence formula is initiated by $b_{-1} = 0$ and $p_0(\lambda) = 1$.

The coefficient $\{a_k\}$ and $\{b_k\}$ will be functions of the moments of ν^2 as, for example

$$a_0 = m_1 \qquad b_0 = \sqrt{m_2 - m_1^2}$$
$$a_1 = \frac{m_3 - 2m_1 m_2 + m_1^3}{m_2 - m_1^2} \qquad b_1 = \cdots$$

Finally, for a given n, the points $\{\lambda_{k,n}\}_{k=1}^n$ are simply the n zeros of $p_n(\lambda)$. The Christoffel-Darboux formula permits to compute the coefficients $\{\gamma_{k,n}\}_{k=1}^n$

$$\gamma_{k,n} = \frac{1}{\sum_{i=0}^{n-1} |p_i(\lambda_{k,n})|^2}.$$

V. Proofs of the Main Results

In Section III, the spectral efficiency of the considered system was said to converge in probability to a deterministic constant when the dimensions K and N of the coding matrices grow indefinitely but with constant ratio $\alpha = K/N$. Furthermore, the limit was expressed in terms of the asymptotic distribution ν^2 of the eigenvalues of the interference matrix $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$. In what follows, we will first discuss the convergence of the empirical eigenvalue distribution ν_N^2 to ν^2 and show how to compute the asymptotic distribution ν^2. Then, we will prove the results stated in Theorems 3.1 and 3.2. Readers that are not interested in mathematical technicalities may skip this section since it is not prerequisite for the following sections.

The results below are based on *free-probability theory* [26], [27], which describes the behavior of random variables defined on noncommutative algebras. In this context, free random variables are the equivalent of independent random variables in classical commutative probability, meaning that the distribution of sums and products of free noncommutative random variables can be expressed in terms of the singular distributions of the original variables.

It is a well-known result of random-matrix theory and free probability that large $N \times N$ independent Hermitian unitarily invariant random matrices can be seen as asymptotic almost sure models for free noncommutative random variables [28]. Now, assume that $\mathbf{M}_1$ and $\mathbf{M}_2$ are two such matrices and that the distributions of their eigenvalues (which are real, due to Hermiticity) tend to the measures μ_1 and μ_2, respectively, as $N \to +\infty$. Then, the eigenvalue distribution of $\mathbf{M}_1 + \mathbf{M}_2$ tends to $\mu_1 \boxplus \mu_2$, the free additive convolution of μ_1 and μ_2 [27], [28]. As a consequence of [29, Proposition 3.5], a similar result can be stated for bi-unitarily invariant matrices and the distributions of their singular values (singular law or singular distribution). Let $\tilde{\mu}_1$ and $\tilde{\mu}_2$ be the asymptotic singular laws[5] of two $N \times N$ independent bi-unitarily invariant random matrices $\mathbf{M}_1$ and $\mathbf{M}_2$. Denote by μ_1 and μ_2 their symmetrization, i.e., $\mu_i(B) = \frac{1}{2}(\tilde{\mu}_i(-B) + \tilde{\mu}_i(B))$ for any Borel set B. Then, the symmetrization of the singular law of $\mathbf{M}_1 + \mathbf{M}_2$ tends to $\mu_1 \boxplus \mu_2$.

Now, consider the interference matrix $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$. Denote by ν the asymptotic distribution of the singular values of $\tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$, which is related to the asymptotic eigenvalue distribution of $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$ by the identity

$$\int f(t)\nu^2(\mathrm{d}t) = \int f(t^2)\nu(\mathrm{d}t)$$

for any measurable function $f(\,\cdot\,)$. Since $\tilde{\mathbf{\Psi}} \tilde{\mathbf{C}} = \sum_{l=1}^{L} g_l h_{dl} \mathbf{C}_l$, intuition suggests that the distribution ν may be computed from the singular value distributions of the matrices $\{g_l h_{dl} \mathbf{C}_l\}$.

Unfortunately, the traditional free additive convolution is not helpful in the general case where $K \neq N$, since the matrices $\mathbf{C}_l$ are not square and, thus, not covered by classical free probability results. An analogous theory for rectangular matrices has been developed by Benaych-Georges in [15] (see also [30]). Since these concepts are very recent and probably not widespread through the technical community, we summarize here the main points and refer the interested readers to the cited papers for a more detailed analysis of the topic. Since the matrices $\mathbf{C}_l$ and $\mathbf{C}_l^H$ have the same nonzero singular values, we will only consider the case $\alpha \leq 1$, the extension to the case $\alpha > 1$ being straightforward.

A. Preliminaries

Let us focus on a symmetric distribution μ (recall that a generic distribution $\tilde{\mu}$ can be symmetrized by taking $\mu(B) = \frac{1}{2}(\tilde{\mu}(-B) + \tilde{\mu}(B))$ for any Borel set B) and define the probability measure μ^2 on $\mathbb{R}_+$ to satisfy $\int f(t^2)\mu(\mathrm{d}t) = \int f(t)\mu^2(\mathrm{d}t)$ for any positive measurable function $f(\,\cdot\,)$. The moment generating series of μ^2 is defined as the formal power series $M_{\mu^2}(z) = \sum_{n=1}^{+\infty} m_n z^n$, where $m_n = \int t^n \mu^2(\mathrm{d}t)$ is the nth moment of μ^2.

For a given $\alpha \in (0,1]$, we denote by $H_\mu(z)$ the rectangular Cauchy transform with ratio α of the distribution μ, defined as

$$H_\mu(z) = zT \circ M_{\mu^2}(z) = z\left(\alpha M_{\mu^2}(z) + 1\right)\left(M_{\mu^2}(z) + 1\right) \tag{17}$$

where $\circ$ denotes composition in the ring of the formal power series, and where we introduced the function $T(z) = (\alpha z + 1)(z+1)$. Let us denote by $H_\mu^{-1}(z)$ the formal inverse of the power series $H_\mu(z)$, that is $H_\mu(H_\mu^{-1}(z)) = H_\mu^{-1}(H_\mu(z)) = z$, which exists since $H_\mu(0) = 0$ and $H'_\mu(0) = 1$ (the first derivative of $H_\mu(z)$ computed at $z = 0$).

Similarly we define $U(z)$ as the formal inverse of $T(z) - 1$ (observe that $T(0) = 1$ and $T'(0) = \alpha + 1$, so that $U(z)$ always exists) and write the *rectangular R-transform with ratio* α of μ as

$$C_\mu(z) = U\left(\frac{z}{H_\mu^{-1}(z)} - 1\right) \tag{18}$$

which is well-defined since $z^{-1}H_\mu^{-1}(z)$ is invertible with respect to multiplication ($\frac{1}{z}H_\mu^{-1}(z) = \frac{1}{H'_\mu(0)} + \cdots$). The formal

[5] The empirical distribution of the singular values of a $N \times N$ matrix is defined as in (6), replacing eigenvalues with singular values. Analogously, its limit for $N \to +\infty$ is intended as in (7).

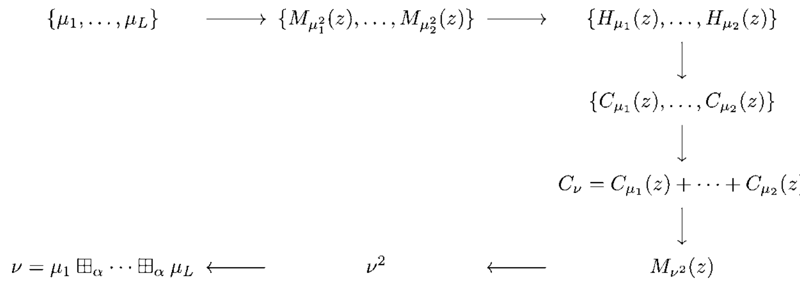

Fig. 1. Algorithm for computing $\nu = \mu_1 \boxplus_\alpha \cdots \boxplus_\alpha \mu_L$.

power series $C_\mu(z)$ identifies unambiguously the underlying probability measure μ.

To recover μ from $C_\mu(z)$, we may proceed as follows. Noting that $U(z)$ is the formal inverse of $T(z) - 1$, from (18) one can write: $\frac{z}{H_\mu^{-1}(z)} = T(C_\mu(z))$. Since $H_\mu(H_\mu^{-1}(z)) = H_\mu^{-1}(H_\mu(z)) = z$, the last equation implies

$$H_\mu(z) = zT[C_\mu(H_\mu(z))]. \tag{19}$$

By comparing this with (17), we readily see that we can compute $M_{\mu^2}(z)$ as the formal power series satisfying

$$M_{\mu^2}(z) = C_\mu[zT(M_{\mu^2}(z))]. \tag{20}$$

It is shown in Appendix A.A that there exists a unique formal power series $M_{\nu^2}(z)$ that is a solution to this equation. Hence (20) completely characterizes $M_{\mu^2}(z)$ from $C_\mu(z)$. Recall that $M_{\mu^2}(z)$ univocally identifies the underlying distribution μ^2 by means of its moments, which can always be computed from (20). Further details are given in Section V-B.

We are now ready to restate ([15, Theorems 3.12 and 3.13]), which are at the basis of the technical results below. First, let us recall that the singular law of a $K \times N$ ($K \leq N$) matrix $\mathbf{X}$ is $\frac{1}{K}\sum_{k=1}^{K} \delta_{\varsigma_k}$, where $\{\varsigma_k\}_{k=1}^{K}$ are the singular values of $\mathbf{X}$. Also, a random matrix is called bi-unitarily invariant if its probability measure is invariant by left- and right-multiplication by constant unitary matrices.

Theorem 5.1 ([15, Theorem 3.13]): Let $K(N)$ be a sequence of integers such that $K(N)/N \to \alpha \in (0, 1]$. Let $\mathbf{X}_N$ and $\mathbf{Y}_N$ be two sequences of $K \times N$ independent random matrices, one of them being bi-unitarily invariant. Assume that the symmetrizations of their singular laws converge in probability towards the probability measures μ_X and μ_Y, respectively. Then, the symmetrization of the singular law of $\mathbf{X}_N + \mathbf{Y}_N$ converges in probability to a new distribution that we denote by $\mu_X \boxplus_\alpha \mu_Y$, the *rectangular-free additive convolution with ratio* α of the measures μ_X and μ_Y.

Using the rectangular R-transform introduced in Section V-A, the resulting distribution can be computed by means of the following theorem.

Theorem 5.2 ([15, Theorem 3.12]): Given $\alpha \in (0, 1]$ and the two symmetric probability measures μ_X and μ_Y on the real line, the function $C_{\mu_X}(z) + C_{\mu_Y}(z)$ is the rectangular R-transform with ratio α of the symmetric probability measure $\mu_X \boxplus_\alpha \mu_Y$. Equivalently

$$C_{\mu_X \boxplus_\alpha \mu_Y} = C_{\mu_X}(z) + C_{\mu_Y}(z).$$

This implies that the binary operator $\boxplus_\alpha$ is commutative and associative.

By associativity, Theorem 5.1 can be readily extended to L (sequences of) bi-unitarily invariant matrices $\mathbf{X}_N^{(1)}, \ldots, \mathbf{X}_N^{(L)}$ with asymptotic singular laws $\mu_1, \ldots, \mu_L$: the symmetrization of the singular law of $\mathbf{X}_N^{(1)} + \cdots + \mathbf{X}_N^{(L)}$ is $\nu = \mu_1 \boxplus_\alpha \cdots \boxplus_\alpha \mu_L$ and can be evaluated following the algorithm summarized in Fig. 1. Now we are ready to prove Theorems 3.1 and 3.2.

B. Proof of Theorem 3.1

Let us denote by ν_N^2 the empirical eigenvalue distribution of $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$, namely $\nu_N^2 = \frac{1}{K}\sum_{k=1}^{K} \delta_{\lambda_k}$, where $\{\lambda_k : k = 1, \ldots, K\}$ are the K positive eigenvalues of the matrix.

Note that $\{\sqrt{\lambda_k} : k = 1, \ldots, K\}$ are the singular values of $\tilde{\mathbf{\Psi}} \tilde{\mathbf{C}} = \sum_{l=1}^{L} g_l h_{dl} \mathbf{C}_l$. According to the isometric coding scheme described in Section II-B, each matrix $\mathbf{C}_l$ is built by extracting K columns of a $N \times N$ ($K < N$) Haar-distributed unitary random matrix. Then, each matrix $g_l h_{dl} \mathbf{C}_l$ is bi-unitarily invariant and the symmetrization of its singular law is $\mu_l = \frac{1}{2}(\delta_{-|g_l h_{dl}|} + \delta_{|g_l h_{dl}|})$, independently of N. Theorem 5.1 implies that the singular law of $\tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$ converges weakly in probability to $\nu = \mu_1 \boxplus_\alpha \cdots \boxplus_\alpha \mu_L$ and, equivalently, that $\nu_N^2 \xrightarrow{\mathrm{P}} \nu^2$ when $K = \alpha N \to +\infty$.

Besides, according to the theory presented in Section V-A, the MGF of ν^2 satisfies the identity

$$M_{\nu^2}(z) = C_\nu[zT(M_{\nu^2}(z))] \tag{21}$$

where $C_\nu(z) = \sum_{l=1}^{L} C_{\mu_l}(z)$ as stated by Theorem 5.2.

Appendix B shows that the rectangular R-transform of μ_l can be expressed as the series expansion of the function

$$C_{\mu_l}(z) = \frac{\sqrt{1 + 4\alpha |g_l h_{dl}|^2 z} - 1}{2\alpha}$$

analytic on $[-(4\alpha|g_l h_{dl}|^2)^{-1}, +\infty)$. This implies that the rectangular R-transform of the distribution ν accepts the analytic representation

$$\begin{aligned} C_\nu(z) &= \sum_{l=1}^{L} C_{\mu_l}(z) \\ &= \frac{1}{2\alpha}\sum_{l=1}^{L}[\sqrt{1 + 4\alpha|g_l h_{dl}|^2 z} - 1] \end{aligned} \tag{22}$$

when $z \geq -(4\alpha \max\{|g_l h_{dl}|^2\})^{-1}$. The coefficients $\{c_i\}$ in Theorem 3.1 are those resulting from the Maclaurin expansion of (22).

By rewriting (21) as

$$M_{\nu^2}(z) = \sum_{i=1}^{+\infty} c_i z^i \left(\alpha M_{\nu^2}^2(z) + (\alpha+1)M_{\nu^2}(z) + 1\right)^i$$

and inserting $M_{\nu^2}(z) = \sum_{i=1}^{+\infty} m_i z^i$, the moments $\{m_i\}$ of ν^2 can be computed by comparing corresponding coefficients of equal powers of z. The first two moments are

$$m_1 = c_1 = \sum_{l=1}^{L} |g_l h_{dl}|^2 \text{ and} \tag{23a}$$

$$\begin{aligned} m_2 &= (\alpha+1)c_1^2 + c_2 \\ &= (\alpha+1)\left(\sum_{l=1}^{L} |g_l h_{dl}|^2\right)^2 - \alpha \sum_{l=1}^{L} |g_l h_{dl}|^4. \end{aligned} \tag{23b}$$

Any symbolic computation software can help in writing the expressions of higher order moments.

C. *Proof of Theorem 3.2*

The asymptotic spectral efficiencies follow directly from the results above. Let γ_N denote the relay contribution to the SINR (3), namely

$$\gamma_N = \mathbf{c}_1^H \tilde{\mathbf{\Psi}}^H \left(\chi \tilde{\mathbf{\Psi}} \mathbf{D}\mathbf{D}^H \tilde{\mathbf{\Psi}}^H + \mathbf{I}_N\right)^{-1} \tilde{\mathbf{\Psi}} \mathbf{c}_1.$$

Recalling that $\tilde{\mathbf{C}} = [\mathbf{c}_1 \; \mathbf{D}]$, the matrix inversion lemma implies that

$$\gamma_N = \frac{\eta_N}{1 - \chi \eta_N}$$

where

$$\eta_N = \mathbf{c}_1^H \tilde{\mathbf{\Psi}}^H \left(\chi \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H + \mathbf{I}_N\right)^{-1} \tilde{\mathbf{\Psi}} \mathbf{c}_1.$$

Let $\mathbf{A}_N = \left(\chi \tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}} + \mathbf{I}_K\right)^{-1} \tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$. Then, the following result holds true.

Proposition 5.1: Consider η_N and $\mathbf{A}_N$ as defined above. Assume that K/N converges to α as N tends to infinity. Then

$$\lim_{N\to+\infty} \left(\eta_N - \frac{1}{K}\mathrm{tr}\{\mathbf{A}_N\}\right) = 0$$

almost surely.

Proof: Since

$$\mathrm{tr}\{\mathbf{A}_N\} = \mathrm{tr}\left\{\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \left(\chi \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H + \mathbf{I}_N\right)^{-1} \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}\right\}$$

the previous result is a direct consequence of the symmetric distribution of the columns of $\tilde{\mathbf{C}}$. The formal proof follows the same guidelines as that of ([18, Proposition 3]) and is thus omitted. ■

Now, the quantity $\frac{1}{K}\mathrm{tr}\{\mathbf{A}_N\}$ can be written in terms of the empirical eigenvalue distribution (6) of $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$ as

$$\frac{1}{K}\mathrm{tr}\{\mathbf{A}_N\} = \int \frac{t}{1+\chi t} \nu_N^2(\mathrm{d}t).$$

Theorem 3.1 tells us that $\nu_N^2 \xrightarrow{\mathrm{P}} \nu^2$ when $K = \alpha N \to +\infty$. Since $\frac{t}{1+\chi t}$ is a bounded function of $t > 0$, we can state that

$$\frac{1}{K}\mathrm{tr}\{\mathbf{A}_N\} \to \int \frac{t}{1+\chi t} \nu^2(\mathrm{d}t) = -\frac{1}{\chi} M_{\nu^2}(-\chi)$$

in probability. The last identity follows from direct comparison with (8). Finally, Proposition 5.1 implies

$$\lim_{K=\alpha N\to+\infty} \eta_N = \eta^{\mathrm{Haar}} = -\frac{1}{\chi} M_\nu^2(-\chi).$$

Note that η^{Haar} is independent of the actual symbol s_k. Then, for $k = 1, \ldots, K$, $\mathrm{SINR}_k \xrightarrow{\mathrm{P}} \mathrm{SINR}^{\mathrm{Haar}}$ as in (10) and $I_{\mathrm{LMMSE}} \xrightarrow{\mathrm{P}} I_{\mathrm{LMMSE}}^{\mathrm{Haar}}$ as in (9) due to continuity of $\gamma_N = \gamma_N(\eta_N)$ and of the logarithmic function.

The asymptotic ML spectral efficiency can be easily derived by recalling that

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln\det(\mathbf{I} + x\mathbf{B}) = \frac{1}{x}\mathrm{tr}\{x\mathbf{B}(\mathbf{I}+x\mathbf{B})^{-1}\}$$

for any square matrix $\mathbf{B}$. Then, the spectral efficiency can be written as

$$\begin{aligned} I_{\mathrm{ML}} &= \frac{\alpha}{1+\alpha} \ln\left(1 + \frac{P_s}{\sigma_d^2}|h_s|^2\right) \\ &\quad + \frac{\alpha}{1+\alpha} \int_{-\chi}^{0} \frac{1}{z} \left[\frac{1}{K}\mathrm{tr}\left\{z\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}} \right.\right. \\ &\quad \left.\left. \times (\mathbf{I}_K - z\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}})^{-1}\right\}\right] \mathrm{d}z \end{aligned}$$

since $\ln\det\mathbf{I} = 0$. The asymptotic spectral efficiency (11) can be obtained[6] by noting that

$$\begin{aligned} &\frac{1}{K}\mathrm{tr}\left\{z\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}(\mathbf{I}_K - z\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}})^{-1}\right\} \\ &\qquad = \frac{1}{K}\sum_{k=1}^{K} \frac{z\lambda_k}{1 - z\lambda_k} = \int \frac{zt}{1-zt} \nu_N^2(\mathrm{d}t) \end{aligned}$$

and that the last expression tends in probability to the moment generating function (8) when $K = \alpha N \to +\infty$, as seen before.

VI. Numerical Illustrations and Simulations

This section gives a numerical assessment of the results above. Summarizing, the presented system behaves (converges in probability to) a deterministic system when the size of the randomly-generated coding matrices grows large keeping constant the coding rate α. The asymptotic spectral efficiency is given by (9) or (11), according to the chosen receiver.

[6]Formally, one should show that the argument of the integral is upper-bounded by a positive integrable function before taking the limit. However, this step is a straightforward consequence of, e.g., Montel's theorem [31] and is therefore omitted.

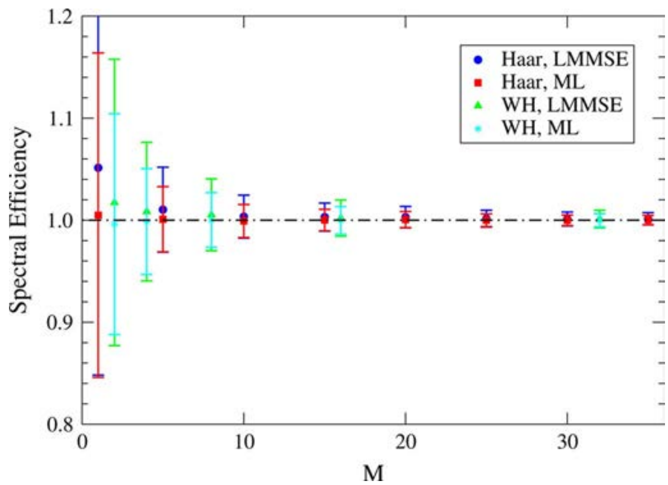

Fig. 2. Simulation results: average spectral efficiency and relative standard deviations. System assumptions: $P_s/\sigma_d^2 = 1$, $h_s = 0$, $L = 2$, $\{|g_l h_{dl}|^2\} = \{1, 1\}$, $\alpha = 3/4$ and $K = 3M$, $N = 4M$. The ordinates are normalized with respect to the asymptotic spectral efficiency at $\alpha = 3/4$, see Fig. 3(a) and (b).

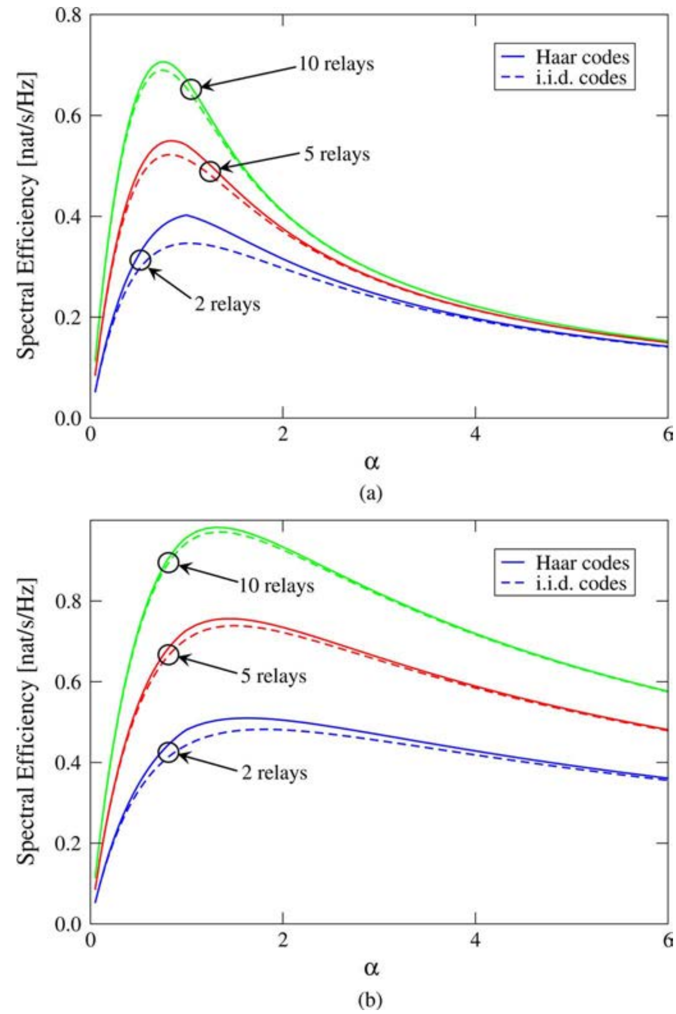

Fig. 3. Spectral efficiency as a function of α for isometric (solid line) and i.i.d. (dashed line) codes. $|h_s|^2 = 0$, $P_s/\sigma_d^2 = 1$ and different numbers of relays, all with unitary channel gains. (a) LMMSE receiver. (b) ML receiver.

Observe that both expressions depend on α only and not on K or N directly. It turns out that these limiting values are excellent approximations of the finite-dimensional codes, even for not-so-large linear-dispersion matrices. To illustrate this, in Fig. 2, we represent the average spectral efficiency over one thousand different realizations of the codes, together with the corresponding standard deviation. All the values are normalized with respect to the asymptotic spectral efficiency. The coding rate α is fixed to 3/4, but the dimensions of the code increase with M, namely $K = 3M$ and $N = 4M$. Note that for $M = 10$, which corresponds to $K = 30$ and $N = 40$, the error is lower than 2%.

Fig. 2 also depicts the performance of a coding scheme based on Walsh-Hadamard matrices, i.e., each linear-dispersion matrix is built as $\mathbf{C}_l = \mathbf{S}_l\mathbf{W}_l$ where the N entries of the diagonal matrix $\mathbf{S}_l$ are i.i.d. 4-PSK symbols and $\mathbf{W}_l$ is made of K randomly-selected columns of an $N \times N$ Walsh-Hadamard matrix. It is evident that the two coding schemes have similar performances, thus suggesting that the analysis presented in this paper can also be used to model the Walsh-Hadamard-based solution. This aspect can be particularly interesting in practical applications, since randomly scrambled Walsh-Hadamard codes are already used in, e.g., UMTS cellular networks. Note however that the proposed solution based on Haar-distributed unitary matrices is more flexible since we can drop the constraint $N = 2^n$, $n = 1, 2, \dots$

Fig. 3(a) depicts the asymptotic LMMSE spectral efficiency (9) as a function of α for different values of the number L of relays. The antiderivative of $M_{\nu^2}(z)/z$, which is needed to depict the asymptotic spectral efficiency $I_{\text{ML}}^{\text{Haar}}$ of the ML receiver (11) in Fig. 3(b), can be straightforwardly computed by means of, e.g., ([32, Formulas 2.261, 2.264-2, and 2.266]). To focus on the effect of the codes, all the curves refer to the case $h_s = 0$.

For both the receivers, the figures also show the asymptotic spectral efficiencies corresponding to the use of i.i.d. codes as obtained from (12). By direct comparison of the two coding schemes, one notices that isometric codes introduce some benefits, as we had anticipated. However, the gain over the i.i.d. scheme—which is around 17% in spectral efficiency (comparing maxima) when considering two relays and the LMMSE filter—decays fast as the number of relays increases. Indeed, Haar codes only cancel the interference generated within each relay; interference among different relays, which becomes predominant when the number of relays increases, is not attenuated by the use of Haar coding matrices. Besides, note that the benefits are less important with the ML receiver (only around 6% with two relays), which is less sensible to colored interference.

The curves in the two graphs also highlight the fact that the coding rate α should be tuned to maximize the spectral efficiency. Unfortunately, analytically locating the maximum is unfeasible, due to the complexity of the expressions involved. The considered situations offer, nevertheless, a clear counterexample that the trivial choice $\alpha = 1$ is not always the best one: maxima can be located both at α lower than 1 (LMMSE example) and at α larger than 1 (ML example). Fig. 4(a) and (b) compare the spectral efficiency achieved by the two coding schemes at their respective optimum coding rate (numerically

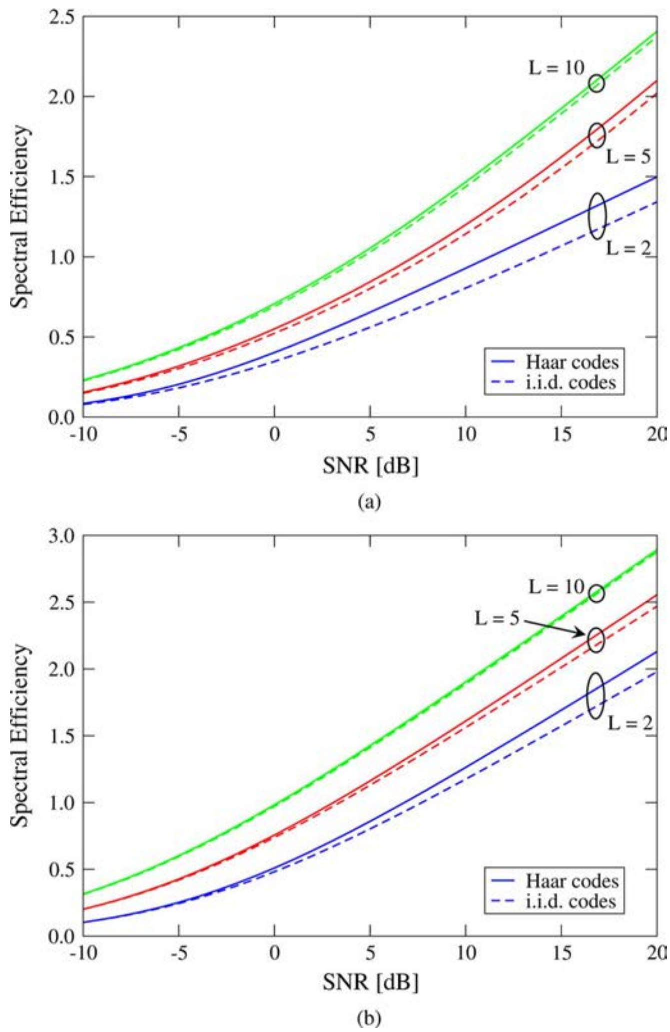

Fig. 4. Spectral efficiency as a function of the SNR P_s/σ_d^2 for isometric (solid line) and i.i.d. (dashed line) codes. $|h_s|^2 = 0$, best α and different numbers of relays, all with unitary channel gains. (a) LMMSE receiver. (b) ML receiver.

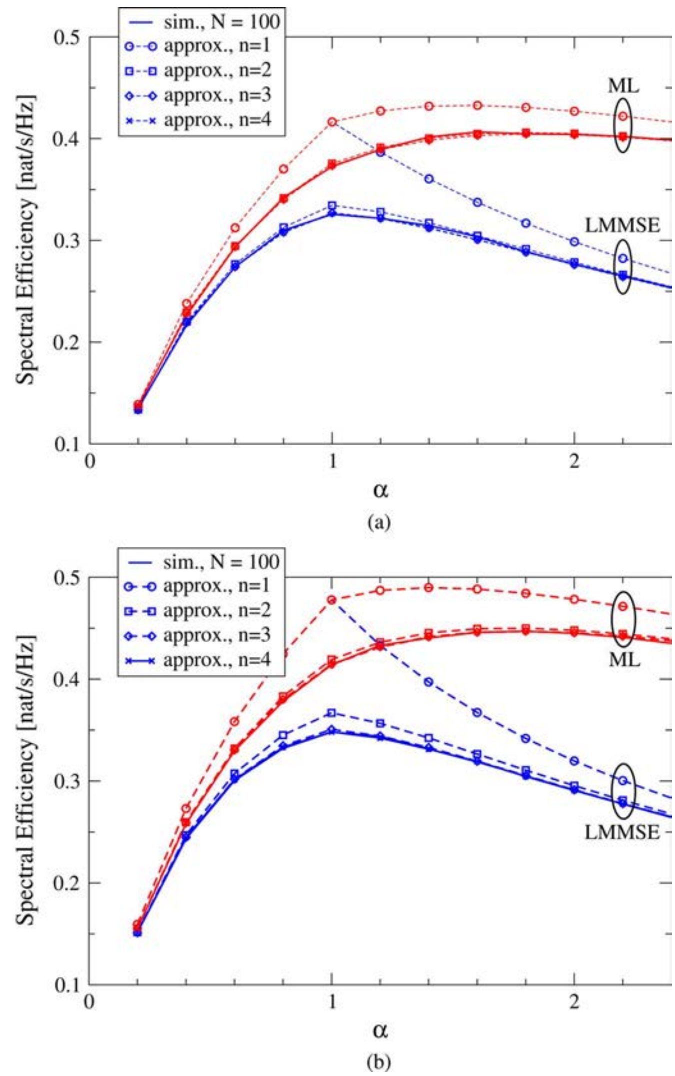

Fig. 5. Comparison between simulation curve and approximations for $n = 1, \ldots, 4$. Systems assumptions: $P_s/\sigma_d^2 = 1$, $|h_s|^2 = 0$ and $N = 100$. Blue curves represent the LMMSE-receiver case, while red curves represent the ML-receiver case. (a) $L = 2, \{|g_l h_{dl}|^2\} = \{0.5, 0.8\}$. (b) $L = 3, \{|g_l h_{dl}|^2\} = \{0.3, 0.5, 0.8\}$.

computed). Once again, one may notice that the maximum gain (around 2 dB) of isometric codes over i.i.d. codes is obtained with two relays and LMMSE receiver.

The moment-based approximation introduced in Section IV.B is validated by comparison with simulation results in Fig. 5, for $L = 2, 3$ and for different values of n. Observe that matching three moments (i.e., $n = 2$) of the asymptotic eigenvalue distribution of the interference matrix $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$ suffices to obtain a good deterministic approximation of a randomly generated code of length $N = 100$, which is realistic in practical applications.

As a final remark, we resume the considerations about the relationship between instantaneous spectral efficiency and outage probability that we started in Section II-A. As mentioned there, further investigation outside the scope of this work is needed for a thorough understanding of the outage behavior of isometric LD-STBC. Indeed, the instantaneous spectral efficiency either is expressed by a very involved formula (case $L = 2$, see (15)) or does not admit a closed form expression (case $L > 2$). However, a rough comparison with the i.i.d. LD-STBC scheme can already be made. Let us consider the outage probability equation in (2) and note, first, the decoding set $\mathcal{L}'$ does not depend on the coding scheme implemented at the relays. Now, according to the results of this paper, the spectral efficiency obtained by a given decoding set is higher for isometric LD-STBC than for i.i.d. LD-STBC. This implies that isometric coding achieves lower outage probability under equal conditions, as it can be observed in Fig. 6, where we reported some simulation results. Equivalently, Fig. 7 depicts the simulated $\varepsilon = 0.1$ outage capacity (more significant in the low-power regime) $C_\varepsilon = \max\{R | P_{\text{out}}(R) < \varepsilon\}$. Unfortunately, at least for these two examples, the gain is not impressive and takes values around 0.5 dB for the LMMSE receiver (and even lower for the ML receiver).

VII. The Low-Power Regime

As mentioned in Section I, probably the main motivation behind the introduction of relays is the desire of achieving high

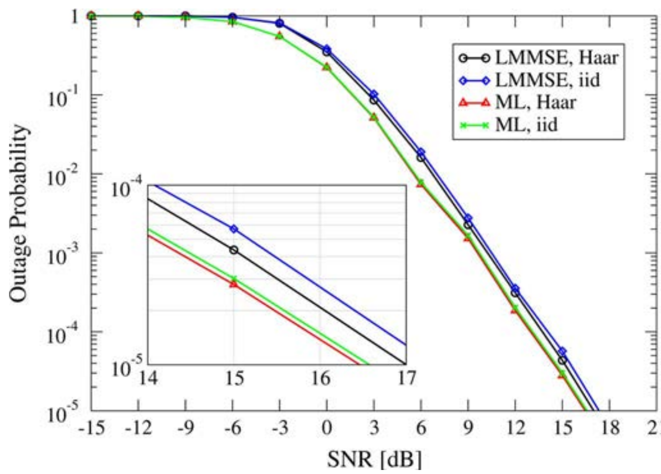

Fig. 6. Outage probability for $L = 2$ relays, unitary channel variances and target transmission rate $R = 0.3$ nat/s/Hz. The coding rate α is fixed to 2.3 for the ML case and to 0.84 for the LMMSE case.

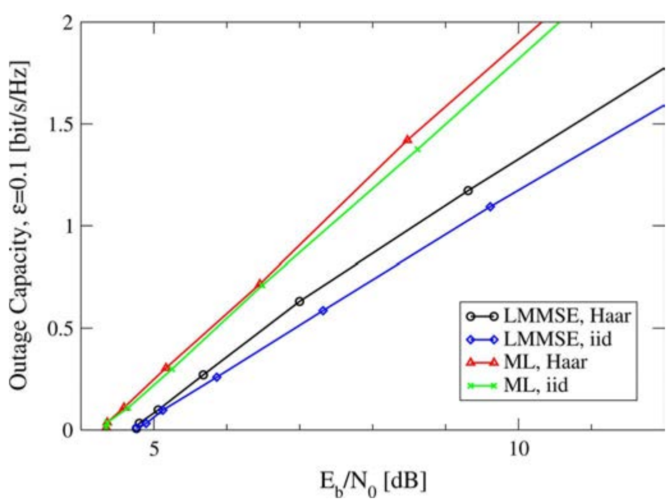

Fig. 7. Outage capacity $C_\varepsilon = \max\{R|P_{\text{out}}(R) < \varepsilon\}$ with $\varepsilon = 0.1$. System parameters are set as in Fig. 6.

data rates by means of distributed space-diversity techniques. However, relays may also be helpful in systems where the received signal-to-noise ratio (SNR) is very low, because of strict energy requirements (e.g., sensor networks) or large source-destination distances (e.g., satellite communications). By improving the quality of the link, relays may reduce power consumption at the source or increase the communications range.

For this reason, in this section we describe the low-power (or wide-band) regime of the considered relay channel. More specifically, we compute the minimum normalized energy per bit that allows reliable transmission, namely [33]

$$\left(\frac{E_b}{N_0}\right)_{\min} = \frac{\ln 2}{\dot{I}(0)} \tag{24}$$

where $\dot{I}(0)$ is the first derivative of the spectral efficiency in the limit for the SNR tending to zero, expressed in nats per degree of freedom. N_0 denotes the noise power spectral density. Besides, as the energy increases from $(E_b/N_0)_{\min}$, the spectral efficiency presents a slope given by [33]

$$S_0 = -\frac{2[\dot{I}(0)]^2}{\ddot{I}(0)} \tag{25}$$

(in bits per degree of freedom per 3 dB), with $\ddot{I}(0)$ being the limit for the SNR tending to zero of the second derivative of the spectral efficiency.

In other words, for the reference SNR $\rho = P_s/\sigma_d^2$ tending to zero, we need to compute the limit of the first- and second-order derivatives of the spectral efficiency. Let m_1 and m_2 be the first two moments of the eigenvalue distribution ν^2 of the interference matrix $\tilde{\mathbf{C}}^H\tilde{\mathbf{\Psi}}^H\tilde{\mathbf{\Psi}}\tilde{\mathbf{C}}$, which are given by (23) in Section V-B. Then, the following results hold true (see also [1]):

Proposition 7.1: Consider the asymptotic spectral efficiencies derived by Theorem 3.2 for isometric LD-STBC. Than, the first two derivatives computed in $\rho = 0$ are

$$\left.\frac{\partial I_{\text{LMMSE}}^{\text{Haar}}}{\partial \rho}\right|_{\rho=0} = \frac{\alpha}{1+\alpha}(|h_s|^2 + m_1) \tag{26a}$$

$$\left.\frac{\partial^2 I_{\text{LMMSE}}^{\text{Haar}}}{\partial \rho^2}\right|_{\rho=0} = -\frac{\alpha}{1+\alpha}(|h_s|^4 + 2|h_s|^2 m_1 + 2m_2 - m_1^2) \tag{26b}$$

for the LMMSE filter and

$$\left.\frac{\partial I_{\text{ML}}^{\text{Haar}}}{\partial \rho}\right|_{\rho=0} = \frac{\alpha}{1+\alpha}(|h_s|^2 + m_1) \tag{27a}$$

$$\left.\frac{\partial^2 I_{\text{ML}}^{\text{Haar}}}{\partial \rho^2}\right|_{\rho=0} = -\frac{\alpha}{1+\alpha}(|h_s|^4 + 2|h_s|^2 m_1 + m_2). \tag{27b}$$

for the ML receiver.

Similarly, the first and second derivatives at $\rho = 0$ of the spectral efficiencies obtained with i.i.d. LD-STBC (see Theorem 3.3) are given by

$$\left.\frac{\partial I^{\text{iid}}}{\partial \rho}\right|_{\rho=0} = \frac{\alpha}{1+\alpha}\left(|h_s|^2 + \sum_{l=1}^{L}|g_l h_{dl}|^2\right)$$

in both cases, and

$$\left.\frac{\partial^2 I_{\text{LMMSE}}^{\text{iid}}}{\partial \rho^2}\right|_{\rho=0} = -\frac{\alpha}{1+\alpha}\left[|h_s|^4 + 2|h_s|^2\sum_{l=1}^{L}|g_l h_{dl}|^2 + (2\alpha+1)\left(\sum_{l=1}^{L}|g_l h_{dl}|^2\right)^2\right]$$

$$\left.\frac{\partial^2 I_{\text{ML}}^{\text{iid}}}{\partial \rho^2}\right|_{\rho=0} = -\frac{\alpha}{1+\alpha}\left[|h_s|^4 + 2|h_s|^2\sum_{l=1}^{L}|g_l h_{dl}|^2 + (\alpha+1)\left(\sum_{l=1}^{L}|g_l h_{dl}|^2\right)^2\right].$$

Inserting these results into (24) and (25), one readily obtains $(E_b/N_0)_{\min}$ and the slopes $S_0^{\text{LMMSE,Haar}}$, $S_0^{\text{ML,Haar}}$, $S_0^{\text{LMMSE,iid}}$, $S_0^{\text{ML,iid}}$.

Proof: See Appendix C. ∎

A. Slope Comparison

Since the four schemes (two possible receivers and two possible codes) present the same minimum energy-per-bit, it is interesting to compare the slopes of the spectral efficiency as E_b/N_0 approaches $(E_b/N_0)_{\min}$ from above. From the expressions of the second-order derivatives, it is straightforward to verify that Haar codes outperform i.i.d. ones for both the receivers. Indeed

$$\frac{S_0^{\text{LMMSE,Haar}}}{S_0^{\text{LMMSE,iid}}} = 1 + \frac{2\alpha \sum_{l=1}^{L} |g_l h_{dl}|^4}{|h_s|^4 + 2|h_s|^2 m_1 + 2m_2 - m_1^2} \quad (28)$$

$$\frac{S_0^{\text{ML,Haar}}}{S_0^{\text{ML,iid}}} = 1 + \frac{\alpha \sum_{l=1}^{L} |g_l h_{dl}|^4}{|h_s|^4 + 2|h_s|^2 m_1 + m_2}. \quad (29)$$

More meaningful is the comparison between the two receivers when employing isometric codes. By replacing the expressions of the second derivatives, one obtains

$$\frac{S_0^{\text{ML,Haar}}}{S_0^{\text{LMMSE,Haar}}} = 1 + \frac{m_2 - m_1^2}{|h_s|^4 + 2|h_s|^2 m_1 + m_2}. \quad (30)$$

It is straightforward to show that

$$1 \leq \frac{S_0^{\text{ML,Haar}}}{S_0^{\text{LMMSE,Haar}}} < 2.$$

Observe that the case $S_0^{\text{ML,Haar}}/S_0^{\text{LMMSE,Haar}} = 1$ arises only when the variance of the distribution ν^2 vanishes, i.e., when $m_2 - m_1^2 = 0$. This condition implies that the matrix $\tilde{\mathbf{C}}^H \tilde{\mathbf{\Psi}}^H \tilde{\mathbf{\Psi}} \tilde{\mathbf{C}}$ is, up to a constant factor, an identity matrix. This is another evidence of the optimality of the LMMSE receiver in the white-interference signal model. Nevertheless, since

$$m_2 - m_1^2 = \alpha \left[\left(\sum_{l=1}^{L} |g_l h_{dl}|^2 \right)^2 - \sum_{l=1}^{L} |g_l h_{dl}|^4 \right]$$

the interference can never be whitened, except for the trivial case $\alpha = 0$.

Note that the three ratios (28), (29), and (30) tend to one as $|h_s|^2$ increases, meaning that all the coding/receiver schemes are equivalent in that situation. The reason is that the relay contribution becomes less important when the quality of the direct link is high.

Fig. 8 compares simulation curves with the approximations derived above, both for the LMMSE receiver [see Fig. 8(a)] and for the ML receiver [see Fig. 8(b)]. The gain of Haar coding over i.i.d. coding is evident. Besides, as commented in Section VI, we can notice once again that Haar signatures are especially useful with the LMMSE receiver, due to higher sensitivity of the linear receiver to colored interference.

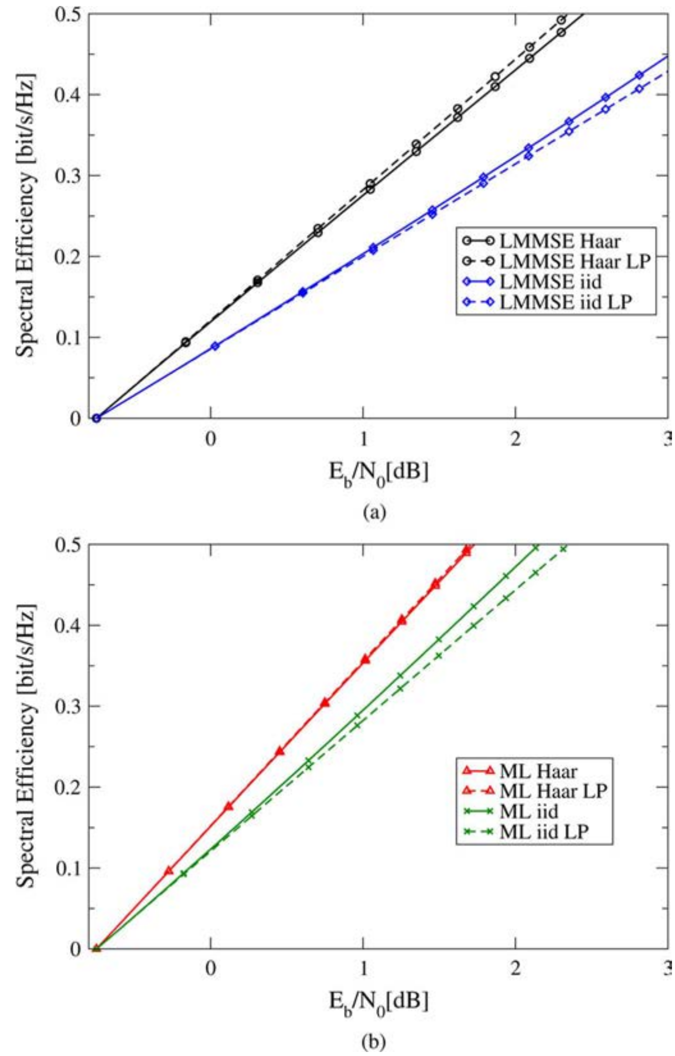

Fig. 8. Spectral efficiency versus E_b/N_0: comparison between real curves and low-power (LP) approximations for the LMMSE (a) and the ML (b) receivers.

VIII. Conclusion

This paper has presented a randomized distributed linear-dispersion space-time block code for the relay channel which is based on isometric matrices. These codes show some gain with respect to similar i.i.d.-based ones [16], [22]. This advantage is due to the orthogonal structure of the coding matrices, which removes intrarelay interference. Intuition and simulation results suggest that isometric codes are more suitable in systems with a low number of relays. Indeed, as we add more terminals, the interference generated within each relay becomes negligible with respect to the one due to the superposition of all relay transmissions. Furthermore, the difference between the two coding schemes is more significant when employing a LMMSE receiver, which is more sensible to colored interference than the ML receiver.

The analysis has been carried out in the asymptotic domain, i.e., when both dimensions of the coding matrices grow indefinitely but keeping constant the coding rate α. Indeed, as in the i.i.d. case, large enough random isometric codes show a deterministic behavior, independent of the specific realization of the matrices. Results have been derived by resorting to the rectangular R-transform, a recent result of probability theory that allows to estimate the distribution of the singular values of a sum of rectangular matrices.

Appendix A
Proofs of Section IV

In this first Appendix we report the proofs of the results in Section IV.

A. *Proof of Lemma 4.1*

First of all, let us prove formally the following result of Section V-A, namely the uniqueness of the solution of (20).

Lemma A.1: Let $C_\nu(z)$ be the rectangular R-transform with ratio α of ν. Then, there exists a unique formal power series $M_{\nu^2}(z)$ that satisfies

$$M_{\nu^2}(z) = C_\nu[zT(M_{\nu^2}(z))]$$

with $T(z) = (\alpha z + 1)(z + 1)$.

Proof: Recalling (17) of the rectangular Cauchy transform with ratio α of ν^2, the fixed point equation can be rewritten as

$$M_{\nu^2}(z) = C_\nu(H_\nu(z)) \tag{31}$$

or, as in (19), $H_\nu(z) = zT[C_\nu(H_\nu(z))]$. Since $C_\nu(0) = 0$ and $T(0) = 1$, the last equation satisfies the assumptions of the Lagrange inversion formula [34], which implies that $H_\nu(z)$ is unique. From (31), and using the fact that $C_\nu(z)$ is invertible by composition, we see that $M_{\nu^2}(z)$ is also unique. ∎

By definition, we know that the rectangular R-transform of ν is the sum of the rectangular R-transforms of the original distributions $\{\mu_l\}$, namely $C_\nu(z) = \sum_{l=1}^L C_{\mu_l}(z)$. We assume now that there exist L functions $M_l(z)$ such that $M_{\nu^2}(z) = \sum_{l=1}^L M_l(z) = \sum_{l=1}^L C_{\mu_l}[zT(\sum_{l=1}^L M_l(z))]$, where the second equality yields from (20). Furthermore, they are solutions to the following system of equations:

$$\begin{bmatrix} M_1(z) \\ \vdots \\ M_L(z) \end{bmatrix} = \begin{bmatrix} C_{\mu_1}\left[z\left(1+\alpha\sum_{l=1}^L M_l(z)\right)\left(1+\sum_{l=1}^L M_l(z)\right)\right] \\ \vdots \\ C_{\mu_L}\left[z\left(1+\alpha\sum_{l=1}^L M_l(z)\right)\left(1+\sum_{l=1}^L M_l(z)\right)\right] \end{bmatrix}.$$

It is simple to prove that this system has a unique solution. It is enough to notice that $M_l(z) = C_{\mu_l}(H_\nu(z))$ and then each of the previous equations can be written as

$$H_\nu(z) = z\left(1+\alpha\sum_{l=1}^L C_{\mu_l}(H_\nu(z))\right)\left(1+\sum_{l=1}^L C_{\mu_l}(H_\nu(z))\right).$$

Similarly to the proof of Lemma A.1,

$$\phi(z) = \left(1+\alpha\sum_{l=1}^L C_{\mu_l}(z)\right)\left(1+\sum_{l=1}^L C_{\mu_l}(z)\right)$$

satisfies the hypotheses of the Lagrange inversion formula, which implies once again that $H_\nu(z)$ and, thus, $M_l(z) = C_{\mu_l}(H_\nu(z))$ exist and are unique.

We can now apply the transformation $x \to x(1+\alpha x)$ to both sides of the system and obtain the equivalent identity

$$\begin{bmatrix} M_1(z)(1+\alpha M_1(z)) \\ \vdots \\ M_L(z)(1+\alpha M_L(z)) \end{bmatrix} = z\left(1+\alpha\sum_{l=1}^L M_l(z)\right) \times \left(1+\sum_{l=1}^L M_l(z)\right)\begin{bmatrix} |g_1 h_{d1}|^2 \\ \vdots \\ |g_L h_{dL}|^2 \end{bmatrix} \tag{32}$$

where the right-hand side (RHS) has been simplified knowing that $\alpha C_{\mu_l}^2(z) + C_{\mu_l}(z) = |g_l h_{dl}|^2 z$ (see (36) in Appendix B). Note that this transformation can introduce solutions. However, only one set $\{M_l(z)\}$ will generate a valid MGF as stated by Lemma A.1.

Now, consider $M_{\nu^2}(z)$ as a function of z on the negative real axis $\mathbb{R}_-$. From its analytic form (8), it is straightforward to prove that $M_{\nu^2}(z)$ is monotonically increasing and bounded between the values -1 and zero. This fact implies that each function $M_l(z)$ is negative and lower-bounded by -1. Indeed, for each individual equation of (32), i.e.

$$M_l(z)(1+\alpha M_l(z)) = z(1+\alpha M_{\nu^2}(z))(1+M_{\nu^2}(z))|g_l h_{dl}|^2$$

one realizes that the RHS is always negative (recall that we consider $\alpha \le 1$). Then, it must be $M_l(z) \in (-1/\alpha, 0]$. Now, since $M_l(z) \le 0$, the equality $M_{\nu^2}(z) = \sum_{l=1}^L M_l(z)$ implies that $-1 < M_l(z) \le 0$. Thus, within the solutions of (32), there must exist a set of functions $\{M_l(z) : l = 1, \ldots, L\}$ such that $-1 < M_l(z) \le 0$ and $-1 < \sum_{l=1}^L M_l(z) \le 0$. Then, $M_{\nu^2}(z) = \sum_{l=1}^L M_l(z)$ is the desired moment generating function.

B. *Proof of Proposition 4.1*

For $L = 2$ relays, the system in (14) can be solved as follows. To simplify the notation, we make the dependence on z implicit and write $M_l = M_l(z)$, $l \in \{1, 2\}$, and $M = M_{\nu^2}(z)$. Furthermore, we denote $\gamma^{(+)} = |g_1 h_{d1}|^2 + |g_2 h_{d2}|^2$ and $\gamma^{(-)} = |g_1 h_{d1}|^2 - |g_2 h_{d2}|^2$. Then, the system of equations can be written as

$$\begin{cases} M_1(1+\alpha M_1) &= z(1+\alpha M)(1+M)|g_1 h_{d1}|^2 \\ M_2(1+\alpha M_2) &= z(1+\alpha M)(1+M)|g_2 h_{d2}|^2. \end{cases} \tag{33}$$

By subtracting the two equations and recalling that we must have $1 + \alpha M \neq 0$, we get

$$M_1 - M_2 = z(1+M)\gamma^{(-)}. \tag{34}$$

On the other hand, adding the two equations of (33) leads to the new identity

$$M - \frac{1}{2}\left(M^2 + (M_1 - M_2)^2\right) = z(1+\alpha M)(1+M)\gamma^{(+)}.$$

By inserting (34), we get the following second-order equation in M:

$$\alpha\left[1 + \left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)}\right]M^2 + 2\left[1 + \alpha\left(z\gamma^{(-)}\right)^2 - z(1+\alpha)\gamma^{(+)}\right]M + \alpha\left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)} = 0$$

which has the two solutions $M \in \{M^{(+)}, M^{(-)}\}$ given by (35), shown at the bottom of the page. Basic algebra shows that the discriminant is positive, meaning that the two solutions exist and are different to one another. However, since

$$\frac{\left[1 + \alpha\left(z\gamma^{(-)}\right)^2 - z(1+\alpha)\gamma^{(+)}\right]}{\alpha\left[1 + \left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)}\right]} > 1$$

for $z < 0$, one has $M^{(+)} < -1$ (the second factor of the RHS of (35) is also larger than one when the plus sign is chosen) and has to be discarded. On the contrary, it is trivial to show that $M^{(-)} \in (-1, 0]$, meaning that the moment generating function is (15).

C. Proof of Proposition 4.2

When all the equivalent channel gains are equal, i.e., $|g_l h_{dl}|^2 = 1$, $l = 1, \ldots, L$, all the linear-dispersion matrices have the same (symmetrized) singular value distribution $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. Then, (14) reduces to

$$\frac{1}{L}M_{\nu^2}\left(1 + \frac{\alpha}{L}M_{\nu^2}\right) = z(1+\alpha M_{\nu^2})(1+M_{\nu^2})$$

since, for any $l = 1, \ldots, L$, $M_l(z) = \frac{1}{L}M_{\nu^2}(z) = C_\mu(H_\nu(z))$ (see also the previous Appendix).

After some algebra, we can write the second-order equation

$$\frac{\alpha}{L}(1 - L^2 z)M_{\nu^2}^2 + (1 - \alpha L z - L z)M_{\nu^2} - Lz = 0.$$

Then, $M_{\nu^2}(z)$ as in (16) is the unique solution that satisfies all the constraints.

Note that in this case one can also solve directly (20) in the real analytic domain, with

$$C_\nu(z) = LC_\mu(z) = \frac{L}{2\alpha}\left[\sqrt{1 + 4\alpha z} - 1\right]$$

defined for $z \geq -\frac{1}{4\alpha}$ (Appendix B shows how to compute the rectangular R-transform $C_\mu(z)$ of $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$). However, it is important to remark that this approach is not always feasible in the general case. Indeed, due the constraint $z \geq -\left(4\alpha \max\{|g_l h_{dl}|^2\}\right)^{-1}$ on the general $C_\nu(z)$ as in (22), identity (20) may not be satisfied at all $z < 0$.

Appendix B
The Rectangular R-Transform of δ_a

According to the algorithm depicted in Fig. 1, we compute here the rectangular R-transform with ratio α corresponding to the symmetrized distribution $\mu = \frac{1}{2}\delta_{-\sqrt{a}} + \delta_{\sqrt{a}}$, that is μ^2 is the distribution of the deterministic constant $a > 0$.

First, the moment generating series $M_{\mu^2}(z) = \sum_{i=1}^{+\infty} a^i z^i$ may be written as $M_{\mu^2}(z) = \frac{az}{1-az}$, which implies $H_\mu(z) = z[1 - (1-\alpha)az]/(1-az)^2$, according to (17). Recalling that $T(z) = (\alpha z + 1)(z+1)$ and that $z = T(U(z)) - 1$, from (18) we know that $C_\mu(z)$ is a solution to

$$H_\mu^{-1}(z) = \frac{z}{(1+C_\mu)(1+\alpha C_\mu)}$$

or, equivalently, to

$$z = H\left(\frac{z}{(1+C_\mu)(1+\alpha C_\mu)}\right) = z\frac{(1+C_\mu)(1+\alpha C_\mu) - (1-\alpha)az}{[(1+C_\mu)(1+\alpha C_\mu) - az]^2}.$$

The last identity can be rewritten as

$$[(1+C_\mu)(1+\alpha C_\mu) - az]^2 = (1+C_\mu)(1+\alpha C_\mu) - (1-\alpha)az$$

and, after some algebra, as

$$[(2\alpha C_\mu + 1 + 2\alpha)^2 - (1 + 4\alpha a z)] \times [(2\alpha C_\mu + 1)^2 - (1 + 4\alpha a z)] = 0.$$

Since it must be $C_\mu(0) = 0$, the first term can be discarded and $C_\mu(z)$ is a solution to $(2\alpha C_\mu + 1)^2 = 1 + 4\alpha a z$, and, thus, to

$$\alpha C_\mu^2 + C_\mu - az = 0. \quad (36)$$

When $z \in \mathbb{R}$ and $z \geq -(4\alpha a)^{-1}$, $C_\mu(z)$ is given by

$$C_\mu(z) = \frac{\sqrt{1 + 4\alpha a z} - 1}{2\alpha}.$$

$$\left\{M^{(+)}, M^{(-)}\right\} = -\frac{\left[1 + \alpha\left(z\gamma^{(-)}\right)^2 - z(1+\alpha)\gamma^{(+)}\right]}{\alpha\left[1 + \left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)}\right]}\left[1 \pm \sqrt{1 - \alpha\frac{\left[\alpha\left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)}\right]\left[1 + \left(z\gamma^{(-)}\right)^2 - 2z\gamma^{(+)}\right]}{\left[1 + \alpha\left(z\gamma^{(-)}\right)^2 - z(1+\alpha)\gamma^{(+)}\right]^2}}\right]. \quad (35)$$

Appendix C
Proof of Proposition 7.1

The results of Proposition 7.1 can be proven as follows. Recall that the moment generating series of ν^2 is $M_{\nu^2}(z) = \sum_{i=1}^{+\infty} m_i z^i$. Now, since $\rho \to 0$ implies $\chi = \rho - |h_s|^2\rho^2 + o(\rho^2)$, one has

$$\eta^{\text{Haar}} = -\frac{1}{\chi} M_{\nu^2}(-\chi) = m_1 - m_2\rho + o(\rho)$$

and, after some algebra

$$\begin{aligned} I_{\text{LMMSE}}^{\text{Haar}} &= \frac{\alpha}{1+\alpha}\Bigg[(|h_s|^2 + m_1)\rho + (m_1^2 - m_2)\rho^2 \\ &\quad -\frac{1}{2}(|h_s|^2 + m_1)^2\rho^2\Bigg] + o(\rho^2) \\ \frac{\partial I_{\text{ML}}^{\text{Haar}}}{\partial \rho} &= \frac{\alpha}{1+\alpha}\frac{|h_s|^2}{1+\rho|h_s|^2} - \frac{\alpha}{1+\alpha}\frac{M_{\nu^2}(-\chi)}{\chi}\frac{\partial \chi}{\partial \rho} \\ &= \frac{\alpha}{1+\alpha}[|h_s|^2 + m_1 \\ &\quad - (|h_s|^4 + 2|h_s|^2 m_1 - m_2)\rho] + o(\rho) \end{aligned}$$

all for ρ small enough. The results in (26) and (27) follow from inspection once recalling the general Maclaurin expansion

$$I(\rho) = I(0) + \sum_{i=1}^{+\infty} \frac{1}{i!}\left(\left.\frac{\partial^i I}{\partial \rho^i}\right|_{\rho=0}\right)\rho^i.$$

Similar reasoning holds for the i.i.d. case.

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