Spectral representations of weakly stationary processes valued in a separable Hilbert space : a survey with applications on functional time series

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Abstract

In this paper, we review and clarify the construction of a spectral theory for weakly-stationary processes valued in a separable Hilbert space. We emphasize the link with functional analysis and provide thorough discussions on the different approaches leading to fundamental results on representations in the spectral domain. The clearest and most complete way to view such representations relies on a Gramian isometry between the time domain and the spectral domain. This theory is particularly useful for modeling functional time series. In this context, we define time invariant operator-valued linear filters in the spectral domain and derive results on composition and inversion of such filters. The advantage of a spectral domain approach over a time domain approach is illustrated through the construction of a class of functional autoregressive fractionaly integrated moving average processes which extend the celebrated class of ARFIMA processes that have been widely and successfully used to model univariate time series. Such functional ARFIMA processes are natural counterparts to processes defined in the time domain that were previously introduced for modeling long range dependence in the context of functional time series.

1 Introduction

Functional data analysis has become an active field of research in the recent decades due to technological advances which makes it possible to store data at very high frequency (and can be considered as continuous time data i.e. functions) or very complex type of data which could be represented by abstract mathematical structures, typically Hilbert spaces. In this framework, we are considering data belonging in a separable Hilbert space which is often taken as the function space $L^2([0,1])$ of square-integrable functions on $[0,1]$. Naturally, researchers on the topic have been interested in generalizing multivariate data analysis and statistical tools to this framework such as inference, estimation, regression, classification or asymptotic results (see, for example, [32], [15]). As for multivariate data, different tools are used when the data are considered independent or not. In this paper, we are interested in functional data with time dependence (functional stochastic processes), that is we observe a family $(X_t)_{t\in \mathrm{T}}$ of random variables where T is a set of index (mainly $\mathbb{Z}$ or $\mathbb{R}$) where for each $t \in \mathrm{T}$, X_t is a random variable from a measurable space $(\Omega, \mathcal{F})$ to a separable Hilbert space $\mathcal{H}_0$ (endowed with its Borel σ-field). In the following, we add the assumption (and give a definition) of weak-stationarity. Examples of such processes are functional linear processes like functional AR or, more generally, functional ARMA processes (see [5, 36, 23]). In the univariate and multivariate (finite-dimensional) cases, spectral analysis of weakly-stationary processes has shown many advantages (see e.g. [6]). Such an analysis has been recently

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popularized in [31, 30, 38] for the functional (infinite-dimensional) framework. In particular, the authors define a spectral representation for weakly stationary functional processes based on the spectral density operator. Existence of such a density is shown under strong assumptions on the autocovariance structure of the process (see the discussion in Section 6.3).

The main goals of this paper are twofold : 1) provide a spectral representation for any weakly stationary processes valued in a general (infinite-dimensional) separable Hilbert space, thus relaxing the assumptions of [31, 30, 38]. 2) derive easy to use results on the composition and inversion of shift-invariant linear filters on such processes. The first point is done following earlier works [21, 27, 20] which generalize multivariate approaches [28, 41, 33]. As far as we know, the second point has not been as explicitly studied before.

Let us recall the classical spectral representation of univariate weakly stationary time series, which goes back to [24] (see also [19] for a survey). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote by $L^2(\Omega, \mathcal{F}, \mathbb{P})$ the space of squared integrable $\mathbb{C}$-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. This space is a Hilbert space when endowed with the inner product $(X, Y) \mapsto \mathbb{E}\left[X\overline{Y}\right]$, where $\overline{Y}$ is the conjugate of Y. Throughout the paper, we moreover let $(\mathrm{T}, +)$ be a locally compact Abelian (l.c.a.) group, whose null element is denoted by 0 (see Appendix B for details).

Definition 1.1 ((Univariate) weakly stationary process)**.** *We say that $X = (X_t)_{t\in\mathrm{T}}$ is a weakly-stationary process if the following assertions hold.*

(i) For all $t \in \mathrm{T}$, $X_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$. We say that X is an L^2 process.

(ii) There exists $\mu \in \mathbb{C}$, called the mean *of X, such that for all $t \in \mathrm{T}$, $\mathbb{E}[X_t] = \mu$. We moreover say that X is* centered *if $\mu = 0$.*

(iii) There exists $\gamma_X : \mathrm{T} \to \mathbb{C}$, called the autocovariance function *of X, such that for all $t, h \in \mathrm{T}$, $\mathrm{Cov}\,(X_{t+h}, X_t) = \gamma_X(h)$.*

We moreover assume that

(iv) the autocovariance function γ_X is continuous on T.

Without loss of meaningful generality, we will only consider centered processes in the following. Condition (iii) simply says that the covariance of the process is shift invariant ((X_s, X_t) and (X_{s+h}, X_{t+h}) have the same covariance for all $s, t, h \in \mathrm{T}$). The continuity condition (iv) is equivalent to say that X is L^2-continuous, and it always holds when $\mathrm{T} = \mathbb{Z}$. As noted in [24, 19], the analysis of centered, weakly-stationary processes if closely linked to functional analysis and, in particular, to unitary representations.

Definition 1.2 ((Continuous) Unitary representations)**.** *Let $(\mathrm{T}, +)$ be an l.c.a. group and $\mathcal{H}_0$ a Hilbert space. A mapping $U : \begin{array}{ccc} \mathrm{T} & \mapsto & \mathcal{L}_b(\mathcal{H}_0) \\ t & \mapsto & U_t \end{array}$ is said to be a* unitary representation *(u.r.) of* T *on $\mathcal{H}_0$ if it satisfies the two following assertions.*

(i) For all $h \in \mathrm{T}$, U_h is a unitary operator from $\mathcal{H}_0$ to $\mathcal{H}_0$.

(ii) The operator U_0 is the identity operator on $\mathcal{H}_0$, that is, $U_0 = \mathrm{Id}_{\mathcal{H}_0}$, and, for all $s, t \in \mathrm{T}$, $U_{s+t} = U_s U_t$.

We say that U is a continuous unitary representation *(c.u.r.) if it moreover satisfies*

(iii) The mapping $h \mapsto U_h$ is continuous on T *for the weak operator topology (w.o.t., that is for all $u, v \in \mathcal{H}_0$, $h \mapsto \langle U_h u, v\rangle_{\mathcal{H}_0}$ is continuous).*

Remark 1.1. *Note that a mapping valued in the set of unitary operators is continuous for the w.o.t. if and only if it is continuous for the strong operator topology (s.o.t., that is for all $u \in \mathcal{H}_0$, $h \mapsto U_h u$ is continuous). Hence, a c.u.r. is continuous for the s.o.t. as a consequence of (iii).*

Let $\mathcal{H}$ be the sub-Hilbert space of centered variables in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and $X = (X_t)_{t\in\mathrm{T}} \in \mathcal{H}^{\mathrm{T}}$ be a centered L^2 process. Denote by

$$\mathcal{H}^X := \overline{\mathrm{Span}}^{\mathcal{H}}\,(X_t, t \in \mathrm{T})$$

the sub-Hilbert space generated by $\{X_t, t \in \mathrm{T}\}$, where the notation $\overline{\mathrm{Span}}^{\mathcal{H}}\,(A)$ means the closure in $\mathcal{H}$ of $\mathrm{Span}\,(A)$. Let U_h^X, $h \in \mathrm{T}$, denote the shift operators defined on $\mathcal{H}^X$ by $U_h^X X_t = X_{t+h}$ for all $t \in \mathrm{T}$. The simple remarks made above about Assertions (iii) and (iv) in Definition 1.1 and Definition 1.2 easily yield the following characterization of weak stationarity.

Lemma 1.1. *Let $X = (X_t)_{t\in\mathrm{T}}$ be a centered L^2 process. Then X is weakly stationary if and only if U^X is a c.u.r. of* T *on $\mathcal{H}^X$.*

Let $\hat{\mathrm{T}}$ denote the dual group of T (the continuous characters defined on T, see Appendix B), and denote by $\mathcal{B}(\hat{\mathrm{T}})$ its Borel σ-field. Under the above assumptions, both γ_X (as a $\mathbb{C}$-valued function on T) and X (as an $\mathcal{H}$-valued function on T) admit spectral counterparts, the first one in the form of a finite non-negative regular measure on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$ and the second one in the form of a countably additive orthogonally scattered (c.a.o.s.) measure on the same space (see Appendix C.2). More precisely, the following theorem holds.

Theorem 1.2 (Spectral measure and spectral representation of a univariate weakly stationary process)**.** *Let $X = (X_t)_{t\in\mathrm{T}}$ be a centered weakly-stationary process with autocovariance function γ_X. Then there exists a unique finite, non-negative, regular measure ν_X on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$, called the* spectral measure *of X, such that*

$$\gamma_X(h) = \int_{\hat{\mathrm{T}}} \chi(h)\, \nu_X(\mathrm{d}\chi), \quad h \in \mathrm{T}\,. \tag{1.1}$$

Moreover, there exists a unique $\mathcal{H}$-valued regular c.a.o.s. measure $\hat{X}$ on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$ such that for all $t \in \mathrm{T}$,

$$X_t = \int_{\hat{\mathrm{T}}} \chi(t)\, \hat{X}(\mathrm{d}\chi)\,, \tag{1.2}$$

and the intensity measure of $\hat{X}$ is ν_X, which means that

$$\mathrm{Cov}\left(\hat{X}(A), \hat{X}(B)\right) = \nu_X(A \cap B) \quad \textit{for all } A, B \in \mathcal{B}(\hat{\mathrm{T}}). \tag{1.3}$$

The identity (1.1) is known as Bochner's theorem. The most commonly used index sets are $\mathrm{T} = \mathbb{Z}$ (discrete time) and $\mathrm{T} = \mathbb{R}$ (continuous time). In the first case, $\mathrm{T} = \mathbb{Z}$ and $\hat{\mathrm{T}} = \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ and the identity (1.1) is then known as Herglotz's theorem.

Note that (1.3) can be rewritten as

$$\mathbb{E}\left[\hat{X}(A)\overline{\hat{X}(B)}\right] = \int \mathbb{1}_A\, \overline{\mathbb{1}_B}\, \mathrm{d}\nu_X\,,$$

hence as saying that $\mathbb{1}_A \mapsto \hat{X}(A)$ is isometric from $L^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \nu_X)$ to $\mathcal{H} \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$. It follows that Relation (1.2) defines the unique isometry I from $L^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \nu_X)$ to $\mathcal{H}$ which maps $(\chi \mapsto \chi(t))$ to X_t for all $t \in \mathrm{T}$. Moreover, this isometry is unitary from $L^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \nu_X)$ to $\mathcal{H}^X$. The former space is called the *spectral domain* of X and the latter its *time domain* and we conclude that the time and spectral domains are isometrically isomorphic. Another consequence of the isometric property of I is that, for all $s, t \in \mathrm{T}$,

$$\mathbb{E}\left[X_s \overline{X_t}\right] = \int_{\hat{\mathrm{T}}} \chi(s-t)\, \nu_X(\mathrm{d}\chi)\,,$$

where we used that, for all $\chi \in \hat{\mathrm{T}}$, $\chi(s)\overline{\chi(t)} = \chi(s-t)$, see [34, Eq. (1) and (6)]. This is exactly (1.1) by setting $h = s - t$. In other words, the results in Theorem 1.2 lead to and are contained in the fact that $L^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \nu_X)$ and $\mathcal{H}^X$ are isometrically isomorphic.

Another consequence of this isometrically isomorphic representation is that we are able to provide a simple definition of linear filtering of weakly-stationary processes in the spectral domain using integration tools for c.a.o.s.measures. Let $\alpha : \hat{\mathrm{T}} \to \mathbb{C}$ measurable and denote by $\mathbb{M}_\alpha$ the set of finite, non-negative regular measures ν on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$ such that $\alpha \in \mathcal{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \nu)$ and by $\mathcal{S}_\alpha$ the set of centered weakly-stationary processes indexed by T whose spectral measure is in $\mathbb{M}_\alpha$. Then, the filter with transfer function α is defined as the mapping

$$F_\alpha : \mathcal{S}_\alpha \to \mathcal{H}^{\mathrm{T}}$$

where for all $X = (X_t)_{t\in\mathrm{T}} \in \mathcal{S}_\alpha$,

$$[F_\alpha(X)]_t = \int_{\hat{\mathrm{T}}} \chi(t)\alpha(\chi)\, \hat{X}(\mathrm{d}\chi), \quad t \in \mathrm{T}\ . \tag{1.4}$$

This means that $Y = F_\alpha(X)$ if and only if Y is weakly stationary and $\mathrm{d}\hat{Y} = \alpha \mathrm{d}\hat{X}$. In this case, it immediately follows that $\mathcal{H}^Y \subset \mathcal{H}^X$, and $\mathrm{d}\nu_Y = |\alpha|^2\, \mathrm{d}\nu_X$. The two following propositions deal with the composition and the inversion of such linear filters. A more general version of them will be stated in Section 3.

Proposition 1.3 (Composition of filters)**.** *Let α and β be measurable functions from $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$ to $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$,*

1. *If $X \in \mathcal{S}_\alpha \cap \mathcal{S}_{\alpha\beta}$, then $F_\alpha(X) \in \mathcal{S}_\beta$ and*
$$F_\beta \circ F_\alpha(X) = F_{\alpha\beta}(X)$$
2. *If $X \in \mathcal{S}_\alpha \cap \mathcal{S}_{\alpha\beta} \cap \mathcal{S}_\beta$, then $F_\alpha(X) \in \mathcal{S}_\beta$, $F_\beta(X) \in \mathcal{S}_\alpha$ and*
$$F_\beta \circ F_\alpha(X) = F_\alpha \circ F_\beta(X) = F_{\alpha\beta}(X)$$

Proposition 1.4 (Inversion of filters)**.** *Let α be a measurable function from $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$ to $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$, $X \in \mathcal{S}_\alpha$ and $Y = F_\alpha(X)$. If $\alpha > 0$ ν_X-a.e. then $\mathcal{H}^Y = \mathcal{H}^X$, $Y \in \mathcal{S}_{1/\alpha}$ and $X = F_{1/\alpha}(Y)$.*

The advantages of describing objects in the spectral domain rather than in the time domain are numerous. Obviously, from a general point of view, a spectral measure is a simpler object than an autocovariance function, and the space $L^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \nu_X)$ is easier to describe than $\mathcal{H}^X$. Similarly, shift-invariant linear filtering is much easier to describe in the spectral domain than in the time domain, in the same way as convolutions of functions of time become pointwise products through the Fourier transform. Composition and inversion of filters can be easily treated as just explained. To conclude this reminder, let us briefly sketch the most direct way to prove Theorem 1.2, following the approach described in [24, 19]. A complete proof is provided in the more general case of Hilbert valued time series, see Theorem 3.2 and its proof in Section 5.2.

Proof of Theorem 1.2 (sketch). As we explained previously, the essential point is to build the unitary mapping between $L^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \nu_X)$ and $\mathcal{H}^X$. To this end, one can rely on the unitary representation provided by the shift operators U_h^X, $h \in \mathrm{T}$, derived in Lemma 1.1. Then Stone's theorem gives that there exists a regular measure ξ^X on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$, valued in the space of orthogonal projections on $\mathcal{H}^X$, such that for all $h \in \mathrm{T}$,

$$U_h^X = \int_{\hat{\mathrm{T}}} \chi(h)\, \xi^X(\mathrm{d}\chi) \; . \tag{1.5}$$

The mapping

$$\hat{X} : \begin{array}{lcl} \mathcal{B}(\hat{\mathrm{T}}) & \to & \mathcal{H} \\ A & \mapsto & \xi^X(A) X_0 \end{array}$$

is then a regular c.a.o.s. measure on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{H})$ and from (1.5) we get

$$X_t = U_t^X X_0 = \int_{\hat{\mathrm{T}}} \chi(t)\, \xi^X(\mathrm{d}\chi) X_0 = \int_{\hat{\mathrm{T}}} \chi(t)\, \hat{X}(\mathrm{d}\chi), \quad t \in \mathrm{T} \; ,$$

which is exactly (1.2). Then, by properties of c.a.o.s. measures this relation defines an isometry and (1.1) comes as a consequence of this result taking for ν_X the intensity measure of $\hat{X}$. □

It is also common to find a proof of Theorem 1.2 where (1.1) is proved first and is used to prove (1.2) (see e.g. [6]). This is a consequence of the close relationship between the functional analysis tools used in the proofs and will be discussed in Section 6.1.

Having recalled the classical univariate case, we can now give more details about the goals of this paper. In the functional case, the space $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ of random variables X defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in a separable Hilbert space $\mathcal{H}_0$ such that $\mathbb{E}\left[\|X\|^2_{\mathcal{H}_0}\right] < +\infty$. We recall that the expectation of $X \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the unique vector $\mathbb{E}[X] \in \mathcal{H}_0$ satisfying

$$\langle \mathbb{E}[X], x \rangle_{\mathcal{H}_0} = \mathbb{E}\left[\langle X, x \rangle_{\mathcal{H}_0}\right], \quad \text{for all } x \in \mathcal{H}_0$$

and that the covariance operator between $X, Y \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is the unique linear operator $\mathrm{Cov}(X, Y) \in \mathcal{L}_b(\mathcal{H}_0)$, satisfying

$$\langle \mathrm{Cov}(X, Y)\, y, x \rangle_{\mathcal{H}_0} = \mathrm{Cov}\left(\langle X, x \rangle_{\mathcal{H}_0}, \langle Y, y \rangle_{\mathcal{H}_0}\right), \quad \text{for all } x, y \in \mathcal{H}_0 \; .$$

In this setting, Definition 1.1 is extended as follow.

Definition 1.3 (Functional weakly stationary process)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $\mathcal{H}_0$ *a separable Hilbert space and* $(\mathrm{T}, +)$ *an l.c.a. group. Then a sequence* $X := (X_t)_{t\in\mathrm{T}}$ *is said to be an* $\mathcal{H}_0$*-valued, weakly-stationary process if*

(i) For all $t \in \mathrm{T}$, $X_t \in L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$.

(ii) For all $t \in \mathrm{T}$, $\mathbb{E}[X_t] = \mathbb{E}[X_0]$. *We say that* X *is centered if* $\mathbb{E}[X_0] = 0$.

(iii) For all $t, h \in \mathrm{T}$, $\mathrm{Cov}\,(X_{t+h}, X_t) = \mathrm{Cov}\,(X_h, X_0)$.

(iv) The autocovariance operator function $\Gamma_X : h \mapsto \mathrm{Cov}\,(X_h, X_0)$ *is weakly continuous i.e. for all* $\Phi \in \mathcal{L}_b(\mathcal{H}_0)$, $h \mapsto \mathrm{Tr}(\Phi\Gamma_X(h))$ *is continuous.*

Given a separable Hilbert space $\mathcal{H}_0$ and a centered weakly stationary $\mathcal{H}_0$-valued process $X := (X_t)_{t\in\mathrm{T}}$, we want to derive

R1 A spectral version of the covariance structure of X similar to (1.1) :

$$\mathrm{Cov}\,(X_s, X_t) = \int_{\hat{\mathrm{T}}} \chi(s-t)\, \nu_X(\mathrm{d}\chi), \quad s, t \in \mathrm{T}\,, \tag{1.6}$$

where ν_X will be called the *spectral operator measure* of X.

R2 A spectral representation of X similar to (1.2) :

$$X_t = \int_{\hat{\mathrm{T}}} \chi(t)\, \hat{X}(\mathrm{d}\chi), \quad \mathbb{P}\text{-a.e.} \quad t \in \mathrm{T}\,, \tag{1.7}$$

as well as a description of the isomorphic relationship that this mapping induces.

R3 A practical definition of shift-invariant linear filters, with results for composition and inversion in the spectral domain.

In [19], the univariate and functional cases are described in a unified setting, by directly considering $(X_t)_{t\in\mathbb{Z}}$ as a $\mathcal{H}$-valued sequence, where $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathbb{P})$ in the univariate case and $\mathcal{H} = L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ in the functional case. However, in the second case, as explained later, $\mathcal{H}$ should be seen as a *normal Hilbert module* rather than just a Hilbert space and this fact has consequences on the previous points, as suggested in the following remarks.

Remark 1.2. *1) About* ***R1*** *: Firstly, since the left hand side term of* (1.6) *is an operator on* $\mathcal{H}_0$ *and for all* $\chi \in \hat{\mathrm{T}}$ *and* $h \in \mathrm{T}$, $\chi(h) \in \mathbb{C}$, *the measure* ν_X *must be operator-valued. Since in the univariate case* ν_X *is a non-negative measure, we expect it to verify an analogous property for the functional case that is to be a* Positive Operator Valued Measure *(p.o.v.m.).*

2) About ***R2*** *: In the univariate case,* $\hat{X}$ *is a measure valued in* $L^2(\Omega, \mathcal{F}, \mathbb{P})$ *and, as discussed above, an advantage of the spectral representation (which is the basis of the general definition of filtering) is to describe linear transformations of the* X_t*'s in the time domain* $\mathcal{H}^X$ *by integrating functions in the spectral domain* $L^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \nu_X)$ *with respect to* $\hat{X}$. *In the functional case, we naturally expect* $\hat{X}$ *to be a measure valued in* $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ *and the spectral domain to be an* L^2 *space related to the spectral operator measure.*

3) About ***R3*** *: In (1.4), one can interpret the filter* F_α *in the spectral domain as a multiplication of* $\hat{X}$ *by a scalar* $\alpha(\chi)$ *depending on the frequency* χ. *Similarly, in the functional case we need to investigate how to apply an* operator $\Phi(\chi)$, *say from* $\mathcal{H}_0$ *to* $\mathcal{G}_0$ *for all frequency* χ, *inside the integral in the right-hand side of (1.7). Then two crucial questions arise:*

a) In which operator space should be valued the $\Phi(\chi)$ *?*

b) Which structure can be associated to the mapping $\Phi \mapsto \int_{\hat{\mathrm{T}}} \Phi\, \mathrm{d}\hat{X}$ *?*

Remark 1.3. *In Definition 1.3, one could have chosen a weaker notion of continuity for the autocovariance operator function, such as continuity for the w.o.t. The necessity of weak-continuity to get* ***R1****,* ***R2****,* ***R3*** *will be made clearer in Section 3 and, in Section 6.2, we will see that, for autocovariance operator functions, weak-continuity is actually equivalent to continuity for the w.o.t.*

The paper is organized as follows. In Section 2 we gather preliminary definitions and results needed all along the paper. In Section 3 we derive precise statements on the spectral representation for functional processes. Then, two applications of these results will be provided in Section 4 to illustrate the advantage of a spectral domain approach over a time domain approach for extending popular univariate time series to the functional case. Proofs are postponed in Section 5, additional comments (including discussion on recent approaches) are made in Section 6 and results on functional analysis and l.c.a. groups are gathered in the appendices.

2 Preliminaries

2.1 Definitions and notation for operator spaces, measurability and L^p spaces

Here we introduce classical definitions for operators on Hilbert spaces (see e.g. [17] for details) and integrals of functions with respect to a measure in the case where the function or the measure is vector-valued (see e.g. [13, Chapter 1] for a nice overview and [12], [11] for a thorough study). This section also contains most of the notation used throughout the paper.

Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces. The inner product and norm, e.g. associated to $\mathcal{H}_0$, are denoted by $\langle \cdot, \cdot \rangle_{\mathcal{H}_0}$ and $\|\cdot\|_{\mathcal{H}_0}$. Let $\mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$ denote the set of linear operators from $\mathcal{H}_0$ to $\mathcal{G}_0$ whose domain (denoted by $\mathcal{D}(\Phi)$) is a linear subspace of $\mathcal{H}_0$, $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ the set of all $\mathcal{H}_0 \to \mathcal{G}_0$ continuous operators. We also denote by $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ the set of all compact operators in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and for all $p \in [1, \infty)$, $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ the Schatten-p class. The space $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and the Schatten-p classes are Banach spaces when respectively endowed with the norms

$$\|\Phi\| := \sup_{\|x\|_{\mathcal{H}_0} \leq 1} \|\Phi x\|_{\mathcal{G}_0} \quad \text{and} \quad \|\Phi\|_p := \left(\sum_{\sigma \in \text{sing}(\Phi)} \sigma^p \right)^{1/p}$$

where $\text{sing}(\Phi)$ is the set of singular values of Φ. Following these definitions, we have, for all $1 \leq p \leq p'$

$$\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{S}_{p'}(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0) \subset \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0). \tag{2.1}$$

The space $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ is endowed with the operator norm and the first three inclusions in (2.1) are continuous embeddings. If $\mathcal{G}_0 = \mathcal{H}_0$, we omit the $\mathcal{G}_0$ in the notations above. As a Banach space, $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ can be endowed with its norm topology but other common topologies are useful. The two most common ones are the strong and weak topologies (respectively denoted by s.o.t. and w.o.t.). We say that a sequence $(\Phi_n)_{n \in \mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)^{\mathbb{N}}$ converges to an operator $\Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ for the s.o.t. if for all $x \in \mathcal{H}_0$, $\lim_{n \to +\infty} \Phi_n x = \Phi x$ in $\mathcal{G}_0$ and for the w.o.t. if for all $x \in \mathcal{H}_0$, for all $y \in \mathcal{G}_0$, $\lim_{n \to +\infty} \langle \Phi_n x, y \rangle_{\mathcal{G}_0} = \langle \Phi x, y \rangle_{\mathcal{G}_0}$.

An operator $\Phi \in \mathcal{L}_b(\mathcal{H}_0)$, is said to be *positive* if for all $x \in \mathcal{H}_0$, $\langle \Phi x, x \rangle_{\mathcal{H}_0} \geq 0$ and we will use the notations $\mathcal{L}_b^+(\mathcal{H}_0)$, $\mathcal{K}^+(\mathcal{H}_0)$, $\mathcal{S}_p^+(\mathcal{H}_0)$ for positive, positive compact and positive Schatten-p operators. If $\Phi \in \mathcal{L}_b^+(\mathcal{H}_0)$ then there exists a unique operator of $\mathcal{L}_b^+(\mathcal{H}_0)$, denoted by $\Phi^{1/2}$, which satisfies $\Phi = \left(\Phi^{1/2}\right)^2$. If Φ is, in addition, compact, then so is $\Phi^{1/2}$. For any $\Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ we denote its adjoint by Φ^{H} (which is compact if Φ is compact). An operator of $\mathcal{L}_b(\mathcal{H}_0)$ is said to be auto-adjoint is it is equal to its adjoint and it is known that any positive operators is auto-adjoint. If $\Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, then $\Phi^H \Phi \in \mathcal{L}_b^+(\mathcal{H}_0)$ and $\Phi \Phi^H \in \mathcal{L}_b^+(\mathcal{G}_0)$ (which are compact if Φ is compact). We define the *absolute value* of Φ as the operator $|\Phi| := \left(\Phi^H \Phi\right)^{1/2} \in \mathcal{L}_b^+(\mathcal{H}_0)$. Moreover, if $\Phi \in \mathcal{S}_1(\mathcal{H}_0)$, $\text{Tr}(\Phi)$ will denote its trace, if $\Phi \in \mathcal{S}_1^+(\mathcal{H}_0)$, it is known that $\text{Tr}(\Phi) = \|\Phi\|_1$. Schatten-1 and Schatten-2 operators are usually referred to as *trace-class* and *Hilbert-Schmidt* operators respectively.

For functions defined on a measurable space $(\mathsf{X}, \mathcal{X})$ and valued in a Banach space $(E, \|\cdot\|_E)$, measurability is defined as follows. A function $f : \mathsf{X} \mapsto E$ is said to be *measurable* if it is the pointwise limit of a sequence of E-valued *simple functions*, i.e. functions belonging in the space $\text{Span}\left(\mathbb{1}_A x \ : \ A \in \mathcal{X},\, x \in E\right)$. When E is separable, this notion is equivalent to the usual Borel-measurability, i.e. to having $f^{-1}(A) \in \mathcal{X}$ for all $A \in \mathcal{B}(E)$, the Borel σ-field on E. We denote by $\mathbb{F}(\mathsf{X}, \mathcal{X}, E)$ (resp. $\mathbb{F}_b(\mathsf{X}, \mathcal{X}, E)$) the space of measurable (resp. bounded measurable) functions from X to E. For a non-negative measure μ and $p \in [1, \infty]$, we denote by $\mathcal{L}^p(\mathsf{X}, \mathcal{X}, E, \mu)$ the space of functions $f \in \mathbb{F}(\mathsf{X}, \mathcal{X}, E)$ such that $\int \|f\|_E^p \, \mathrm{d}\mu$ (or μ- essup $\|f\|_E$ for $p = \infty$) is finite and by $L^p(\mathsf{X}, \mathcal{X}, E, \mu)$ its quotient space with respect to μ-a.e. equality, or, equivalently, with respect to the subspace of functions f such that $f = 0$ μ-a.e., which we write

$$L^p(\mathsf{X}, \mathcal{X}, E, \mu) = \mathcal{L}^p(\mathsf{X}, \mathcal{X}, E, \mu) \Big/ \{f \,:\, f = 0 \ \mu\text{-a.e.}\} \ .$$

The corresponding norms are denoted by $\|f\|_{L^p(\mathsf{X},\mathcal{X},E,\mu)}$. For $p \in [1, \infty)$, the space of simple measurable functions with finite-measure support, i.e. $\mathrm{Span}\left(\mathbb{1}_A x \,:\, A \in \mathcal{X},\, \mu(A) < \infty,\, x \in E\right)$, is dense in $L^p(\mathsf{X}, \mathcal{X}, E, \mu)$. For $f \in \mathrm{Span}\left(\mathbb{1}_A x \,:\, A \in \mathcal{X},\, \mu(A) < \infty,\, x \in E\right)$ with range $\{\alpha_1, \cdots, \alpha_n\}$, the integral (often referred to as the *Bochner integral*) of the E-valued function f with respect to μ is defined by

$$\int f \, \mathrm{d}\mu = \sum_{k=1}^{n} \alpha_k \, \mu\left(f^{-1}(\{\alpha_k\})\right) \in E \ . \tag{2.2}$$

This integral is extended to $L^1(\mathsf{X}, \mathcal{X}, E, \mu)$ by continuity (and thus also to L^p if μ is finite).

An E-valued measure is a mapping $\mu : \mathcal{X} \to E$ such that for any sequence $(A_n)_{n\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$ of pairwise disjoint sets then $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu(A_n)$ where the series converges in E, that is

$$\lim_{N\to+\infty} \left\| \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) - \sum_{n=0}^{N} \mu(A_n) \right\|_E = 0 \ .$$

We denote by $\mathbb{M}(\mathsf{X}, \mathcal{X}, E)$ the set of E-valued measures. For such a measure μ, the mapping

$$\|\mu\|_E : A \mapsto \sup \left\{ \sum_{i\in\mathbb{N}} \|\mu(A_i)\|_E \ :\ (A_i)_{i\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}} \text{ is a countable partition of } A \right\}$$

defines a non-negative measure on $(\mathsf{X}, \mathcal{X})$ called the *variation measure* of μ. The notation $\|\mu\|_E$ will be adapted to the notation chosen for the norm in E (for example if μ is a complex measure we will use $|\mu|$ and if μ is valued in a Schatten-p space we will use $\|\mu\|_p$). The variation of a complex-valued measure is always finite and the variation of a non-negative measure is itself. We will denote by $\mathbb{M}_b(\mathsf{X}, \mathcal{X}, E)$ the set of E-valued measures with finite variation. It is a Banach space when endowed with the norm $\|\mu\|_{TV,E} = \|\mu\|_E(\mathsf{X})$. If $\mu \in \mathbb{M}_b(\mathsf{X}, \mathcal{X}, E)$, then for a simple function $f : \mathsf{X} \mapsto \mathbb{C}$ with range $\{\alpha_1, \ldots, \alpha_n\}$, the integral of f with respect to μ is defined by the same formula as in (2.2) (but this time the α_k's are scalar and the μ's are E-valued). This definition is extended to $L^1(\mathsf{X}, \mathcal{X}, \|\mu\|_E)$ by continuity.

When X is a locally-compact topological space, a vector measure $\mu \in \mathbb{M}(\mathsf{X}, \mathcal{X}, E)$ is said to be *regular* if for all $A \in \mathcal{X}$, for all $\epsilon > 0$, there exist a compact set $K \in \mathcal{X}$ and an open set $U \in \mathcal{X}$ with $K \subset A \subset U$ such that for all $B \in \mathcal{X}$ satisfying $B \subset U \setminus K$, $\|\mu(B)\|_E \leq \epsilon$. We denote by $\mathbb{M}_r(\mathsf{X}, \mathcal{X}, E)$ the linear space of such measures. The notion of regularity is extended to non-finite, non-negative measures by restricting A to be such that $\mu(A) < +\infty$. From the straightforward inequality $\|\mu(A)\|_E \leq \|\mu\|_E(A)$ for all $A \in \mathcal{X}$, we get that if $\mu \in \mathbb{M}_b(\mathsf{X}, \mathcal{X}, E)$ has a regular variation, then μ is regular. The converse is not always true but holds for complex measures. An interesting result (see [20, Remark 3.6.2]) is that an E-valued measure ν is regular if and only if for all $\phi \in E^*$, $\phi \circ \nu$ is a regular complex measure.

Finally, we recall another notion of measurability for functions valued in the operator spaces $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ or $\mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$. Namely, a function $\Phi : \mathsf{X} \to \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ is said to be *simply measurable* if for all $x \in \mathcal{H}_0$, $t \mapsto \Phi(t)x$ is measurable as a $\mathcal{G}_0$-valued function. The set of such functions is denoted by $\mathbb{F}_s\left(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0\right)$. For a function $\Phi : \mathsf{X} \to \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$, adapting [27], [20, Section 3.4], we will say that Φ is $\mathcal{O}$-measurable if it satisfies the two following conditions.

(i) For all $x \in \mathcal{H}_0$, $\{t \in \mathsf{X} \,:\, x \in \mathcal{D}(\Phi(t))\} \in \mathcal{X}$.

(ii) There exist a sequence $(\Phi_n)_{n\in\mathbb{N}}$ valued in $\mathbb{F}_s\left(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0\right)$ such that for all $t \in \mathsf{X}$ and $x \in \mathcal{D}(\Phi(t))$, $\Phi_n(t)x$ converges to $\Phi(t)x$ in $\mathcal{G}_0$ as $n \to \infty$.

We denote by $\mathbb{F}_{\mathcal{O}}\left(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0\right)$ the space of such functions Φ. Note that for all Banach space $\mathcal{E}$ which is continuously embedded in $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ (e.g. $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ for $p \geq 1$ or $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$), the following inclusions hold

$$\mathbb{F}(\mathsf{X}, \mathcal{X}, \mathcal{E}) \subset \mathbb{F}_s\left(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0\right) \subset \mathbb{F}_{\mathcal{O}}\left(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0\right) \ . \tag{2.3}$$

In this paper we will mainly take $\mathcal{E}$ as the set of trace-class, Hilbert-Schmidt or compact $\mathcal{H}_0 \to \mathcal{G}_0$ operators for which measurability and simple measurability are equivalent as stated in the following lemma.

Lemma 2.1. *Let $\mathcal{E} = \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ or $\mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$ where $p \in \{1, 2\}$ and $\mathcal{H}_0, \mathcal{G}_0$ are separable Hilbert spaces. Then a function $\Phi : \mathsf{X} \to \mathcal{E}$ is measurable if and only if it is simply measurable.*

Proof. See Section 5.1. $\square$

We also need to consider operator-valued measures for our study, and more particularly p.o.v.m.'s which are studied in the next section.

2.2 Positive Operator Valued Measures

The notion of Positive Operator Valued Measures is widely used in Quantum Mechanics and a good study of such measures can be found in [4]. Here we provide useful definitions and results for our purpose.

Definition 2.1 (Positive Operator Valued Measures)**.** *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and $\mathcal{H}_0$ be a separable Hilbert space. A Positive Operator Valued Measure (p.o.v.m.) on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ is a mapping $\nu : \mathcal{X} \to \mathcal{L}_b^+(\mathcal{H}_0)$ such that for all sequence of disjoint sets $(A_n)_{n\in\mathbb{N}} \in \mathcal{X}^{\mathbb{N}}$,*

$$\nu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu(A_n) \tag{2.4}$$

where the series converges in $\mathcal{L}_b^+(\mathcal{H}_0)$ for the s.o.t.

Due to properties of positive operators, convergence in the w.o.t. would be sufficient in Definition 2.1, see [4, Proposition 1]. Note that, with this definition, a p.o.v.m. is not a vector-valued measure in the sense of Section 2.1 since we do not suppose that the series in (2.4) converges in operator norm. However, this definition is sufficient to derive a useful characterization which links a p.o.v.m. to a sesquilinear, hermitian, positive semi-definite, continuous mapping valued in $\mathbb{M}(\mathsf{X}, \mathcal{X})$.

Definition 2.2. *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and $\mathcal{H}_0$ be a separable Hilbert space. A mapping $\phi : \mathcal{H}_0^2 \to \mathbb{M}(\mathsf{X}, \mathcal{X})$ is said to be sesquilinear, hermitian, positive semi-definite, continuous if for all $A \in \mathcal{X}$, the mapping $(x, y) \mapsto \phi(x, y)(A)$ is sesquilinear, hermitian, positive semi-definite, continuous.*

The characterization of p.o.v.m.'s then reads as follows (see [4, Theorem 2]).

Proposition 2.2. *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and $\mathcal{H}_0$ be a separable Hilbert space, then the following assertions hold.*

(i) For all p.o.v.m. ν on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ and all $x, y \in \mathcal{H}_0$, the mapping $y^{\mathsf{H}}\nu x : A \mapsto \langle \nu(A)x, y\rangle_{\mathcal{H}_0}$ is a complex-valued measure on $(\mathsf{X}, \mathcal{X})$. Moreover, the mapping $(x, y) \mapsto y^{\mathsf{H}}\nu x$ is sesquilinear, hermitian, positive semi-definite, continuous.

(ii) Conversely, if $\phi : \mathcal{H}_0^2 \to \mathbb{M}(\mathsf{X}, \mathcal{X})$ is a sesquilinear, hermitian, positive semi-definite bounded mapping, then there exists a unique p.o.v.m. ν on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ such that for all $x, y \in \mathcal{H}_0$, $\phi(x, y) = y^{\mathsf{H}}\nu x$.

This characterization can be used to construct integrals of bounded complex-valued functions with respect to p.o.v.m.'s and we refer to [4, Section 5] for details. When X is a locally-compact topological space, this also gives a simple notion of regularity for p.o.v.m.'s, namely a p.o.v.m. ν on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ is said to be *regular* if for all $x, y \in \mathcal{H}_0$, the measure $y^{\mathsf{H}}\nu x$ is a regular complex measure. We will say that a p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ is *trace-class* if it is $\mathcal{S}_1(\mathcal{H}_0)$-valued. The following lemma states that trace-class p.o.v.m.'s can be seen as vector-valued measures.

Lemma 2.3. *A p.o.v.m. ν on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ is trace-class if and only if $\nu(\mathsf{X}) \in \mathcal{S}_1(\mathcal{H}_0)$. In this case, ν is a $\mathcal{S}_1(\mathcal{H}_0)$-valued measure (in the sense that (2.4) holds in $\|\cdot\|_1$-norm) with finite variation measure $\|\nu\|_1 : A \mapsto \|\nu(A)\|_1$. Moreover, regularity of ν as a p.o.v.m. is equivalent to regularity of ν as a $\mathcal{S}_1(\mathcal{H}_0)$-valued measure which is itself equivalent to regularity of $\|\nu\|_1$.*

Proof. See Section 5.1. $\square$

Thanks to this result, integration of complex-valued functions with respect to a trace-class p.o.v.m. is possible using the theory of vector-valued measures with finite variation recalled in Section 2.1. Finally, the following Radon-Nikodym property holds.

Theorem 2.4. *Let $(\mathsf{X}, \mathcal{X})$ be a measure space, $\mathcal{H}_0$ a separable Hilbert space and ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$. Let μ be a finite non-negative measure on $(\mathsf{X}, \mathcal{X})$. Then $\|\nu\|_1 \ll \mu$ (i.e. for all $A \in \mathcal{X}$, $\mu(A) = 0 \Rightarrow \|\nu\|_1(A) = 0$), if and only if there exists $g \in L^1(\mathsf{X}, \mathcal{X}, \mathcal{S}_1(\mathcal{H}_0), \mu)$ such that $\mathrm{d}\nu = g\,\mathrm{d}\mu$, i.e. for all $A \in \mathcal{X}$,*

$$\nu(A) = \int_A g\,\mathrm{d}\mu\ . \tag{2.5}$$

In this case, g is unique and is called the density of ν with respect to μ and denoted as $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Moreover,

(a) For μ-almost every $t \in \mathsf{X}$, $g(t) \in \mathcal{S}_1^+(\mathcal{H}_0)$.

(b) The density of $\|\nu\|_1$ with respect to μ is $\|g\|_1$. In particular, $g = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}\|g\|_1$.

(c) If $\|\nu\|_1 \leq \mu$, then $\|g\|_1 \leq 1$ μ-a.e., and if $\mu = \|\nu\|_1$, then $\|g\|_1 = 1$ μ-a.e.

2.3 Normal Hilbert modules

Modules extend the notion of vector spaces to the case where scalar multiplication is replaced by a multiplicative operation with elements of a ring. When the ring is a C^*-algebra, it is possible to endow a module with a structure similar to a Hilbert space (see [22]). In the following we consider the C^*-algebra $\mathcal{L}_b(\mathcal{H}_0)$ where $\mathcal{H}_0$ is a separable Hilbert space as presented in [20].

Definition 2.3 ($\mathcal{L}_b(\mathcal{H}_0)$-module)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space. A $\mathcal{L}_b(\mathcal{H}_0)$-module is a commutative group $(\mathcal{H}, +)$ such that there exists a multiplicative operation (called the* module action*)*

$$\begin{array}{lcl} \mathcal{L}_b(\mathcal{H}_0) \times \mathcal{H} & \to & \mathcal{H} \\ (\Phi, x) & \mapsto & \Phi \bullet x \end{array}$$

which satisfies the usual distributive properties : for all $\Phi, \Psi \in \mathcal{L}_b(\mathcal{H}_0)$, and $x, y \in \mathcal{H}$,

$$\begin{aligned} \Phi \bullet (x + y) &= \Phi \bullet x + \Phi \bullet y, \\ (\Phi + \Psi) \bullet x &= \Phi \bullet x + \Psi \bullet x, \\ (\Phi\Psi) \bullet x &= \Phi \bullet (\Psi \bullet x), \\ \mathrm{Id}_{\mathcal{H}_0} \bullet x &= x. \end{aligned}$$

Definition 2.4 ((Normal) pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space. A pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module $\mathcal{H}$ is a $\mathcal{L}_b(\mathcal{H}_0)$-module endowed with a mapping $[\cdot,\cdot]_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathcal{L}_b(\mathcal{H}_0)$ satisfying for all $x, y, z \in \mathcal{H}$, and $\Phi \in \mathcal{L}_b(\mathcal{H}_0)$,*

(i) $[x, x]_{\mathcal{H}} \in \mathcal{L}_b^+(\mathcal{H}_0)$,

(ii) $[x, x]_{\mathcal{H}} = 0$ if and only if $x = 0$,

(iii) $[x + \Phi \bullet y, z]_{\mathcal{H}} = [x, z]_{\mathcal{H}} + \Phi[y, z]_{\mathcal{H}}$,

(iv) $[y, x]_{\mathcal{H}} = [x, y]_{\mathcal{H}}^{\mathsf{H}}$.

If moreover, for all $x, y \in \mathcal{H}$, $[x, y]_{\mathcal{H}} \in \mathcal{S}_1(\mathcal{H}_0)$, we say that $[\cdot,\cdot]_{\mathcal{H}}$ is a gramian *and that $\mathcal{H}$ is a* normal *pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module.*

The mapping $[\cdot,\cdot]_{\mathcal{H}}$ generalizes the notion of scalar products for $\mathcal{L}_b(\mathcal{H}_0)$-modules and is often called a $\mathcal{L}_b(\mathcal{H}_0)$-valued scalar product. In the following, we only consider normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules even if some notions can be defined when $[\cdot,\cdot]_{\mathcal{H}}$ is not a gramian. Note that a $\mathcal{L}_b(\mathcal{H}_0)$-module is a vector space if we define the scalar-vector multiplication by $\alpha x = (\alpha \mathrm{Id}_{\mathcal{H}_0}) \bullet x$ for all $\alpha \in \mathbb{C}$, $x \in \mathcal{H}$ and that, in the particular case where $[\cdot,\cdot]$ is a gramian, then $\langle\cdot,\cdot\rangle := \mathrm{Tr}[\cdot,\cdot]$ is a scalar product. Hence a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is also a pre-Hilbert space. If it is complete (for the norm $\|x\|_{\mathcal{H}} = \left\|[x, x]_{\mathcal{H}}\right\|_1^{1/2}$), then it is called a *normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module.* For normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules, the notions of sub-modules and $\mathcal{L}_b(\mathcal{H}_0)$-linear span as well as $\mathcal{L}_b(\mathcal{H}_0)$-linear operators, gramian-isometries, gramian-unitary operators, gramian-orthogonality, gramian-projections come as natural extensions of their vector space counterparts. For completeness, we provide here the necessary definitions and refer to chapter II of [20] for a complete study.

Definition 2.5 (Submodules and $\mathcal{L}_b(\mathcal{H}_0)$-linear operators)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space and $\mathcal{H}, \mathcal{G}$ two $\mathcal{L}_b(\mathcal{H}_0)$-modules. Then a subset of $\mathcal{H}$ is called a* submodule *if it is a $\mathcal{L}_b(\mathcal{H}_0)$-module. An operator $T \in \mathcal{L}_b(\mathcal{H}, \mathcal{G})$ is said to be* $\mathcal{L}_b(\mathcal{H}_0)$-linear *if for all $\Phi \in \mathcal{L}_b(\mathcal{H}_0)$ and $x \in \mathcal{H}$, $T(\Phi \bullet x) = \Phi \bullet (Tx)$.*

Definition 2.6 (Gramian-isometries)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}, \mathcal{G}$ be two pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules and $U : \mathcal{H} \to \mathcal{G}$ a $\mathcal{L}_b(\mathcal{H}_0)$-linear operator. Then U is said to be*

(i) a gramian-isometry *(or gramian-isometric) if for all $x, y \in \mathcal{H}$, $[Ux, Uy]_{\mathcal{G}} = [x, y]_{\mathcal{H}}$,*

(ii) gramian-unitary *if it is a bijective gramian-isometry.*

The space $\mathcal{H}$ is said to be gramian-isometrically-embedded *in $\mathcal{G}$ (denoted $\mathcal{H} \underset{\approx}{\subsetneq} \mathcal{G}$) if there exists a gramian-isometry from $\mathcal{H}$ to $\mathcal{G}$. The spaces $\mathcal{H}$ and $\mathcal{G}$ are said to be* gramian-isomorphic *(denoted $\mathcal{H} \cong \mathcal{G}$) if there exists a gramian-unitary operator from $\mathcal{H}$ to $\mathcal{G}$.*

Definition 2.7 ((Continuous) gramian unitary representations)**.** *Let $(\mathrm{T}, +)$ be an l.c.a. group, $\mathcal{H}_0$ a Hilbert space and $\mathcal{H}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module with gramian $[\cdot, \cdot]_{\mathcal{H}}$. A mapping $U : \begin{array}{ccc} \mathrm{T} & \mapsto & \mathcal{L}_b(\mathcal{H}) \\ t & \mapsto & U_t \end{array}$ is said to be a* gramian unitary representation *(g.u.r.) of T on $\mathcal{H}$ if it is an u.r. of T on $\mathcal{H}$ such that for all $h \in \mathrm{T}$, U_h is gramian-unitary. A g.u.r. is continuous, then called a c.g.u.r., if it is continuous as an u.r.*

For later reference we state a simple extension result for gramian isometric operators.

Proposition 2.5 (Gramian-isometric extension)**.** *Let $\mathcal{H}$ be a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module, $\mathcal{G}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module. Let $(v_j)_{j \in J}$ and $(w_j)_{j \in J}$ be two sets of vectors in $\mathcal{H}$ and $\mathcal{G}$ respectively with J an arbitrary index set. If for all $i, j \in J$, $[v_i, v_j]_{\mathcal{H}} = [w_i, w_j]_{\mathcal{G}}$ then there exists a unique gramian-isometry*

$$S : \overline{\mathrm{Span}}^{\mathcal{H}} (\Phi \bullet v_j, \Phi \in \mathcal{L}_b(\mathcal{H}_0), j \in J) \to \mathcal{G}$$

such that for all $j \in J$, $S v_j = w_j$. If moreover $\mathcal{H}$ is complete then

$$S\left(\overline{\mathrm{Span}}^{\mathcal{H}} (\Phi \bullet v_j, \Phi \in \mathcal{L}_b(\mathcal{H}_0), j \in J)\right) = \overline{\mathrm{Span}}^{\mathcal{G}} (\Phi \bullet w_j, \Phi \in \mathcal{L}_b(\mathcal{H}_0), j \in J)$$

We can now state an important result, which generalizes Stone's theorem to c.g.u.r.'s. We refer to [20, Proposition 2.5.4] for a proof and Appendix C.1 for the definition of gramian-projection valued measures.

Theorem 2.6 (Stone's theorem for modules)**.** *Let $(\mathrm{T}, +)$ be an l.c.a. group, $\mathcal{H}_0$ a Hilbert space, $\mathcal{H}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module with gramian $[\cdot, \cdot]_{\mathcal{H}}$ and $U : \begin{array}{ccc} \mathrm{T} & \mapsto & \mathcal{L}_b(\mathcal{H}) \\ t & \mapsto & U_t \end{array}$ a c.g.u.r of T on $\mathcal{H}$. Then there exists a unique regular gramian-projection valued measure ξ on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$ such that*

$$U_h = \int_{\hat{\mathrm{T}}} \chi(h)\, \xi(\mathrm{d}\chi), \quad h \in \mathrm{T}\,. \tag{2.6}$$

We conclude this section with some examples of normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules.

Example 2.1. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces.*

- *$\mathcal{H}_0$ is itself a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module with module action $\Phi \bullet x = \Phi x$ and gramian $[x, y]_{\mathcal{H}_0} = x \otimes y$ where $(x \otimes y)u = \langle u, y \rangle_{\mathcal{H}_0} x$ for all $u \in \mathcal{H}_0$.*
- *$\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module with module action $\Phi \bullet \Psi = \Phi\Psi$ and gramian $[\Psi, \Theta]_{\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)} = \Psi\Theta^{\mathsf{H}}$.*
- *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and μ a finite non-negative measure on $(\mathsf{X}, \mathcal{X})$. Then for all normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module $\mathcal{H}$, the space $L^2(\mathsf{X}, \mathcal{X}, \mathcal{H}, \mu)$ is a normal $\mathcal{L}_b(\mathcal{H}_0)$-Hilbert module for the module action $(\Phi \bullet f)(\cdot) = \Phi \bullet [f(\cdot)]$ and gramian $[f, g]_{L^2(\mathsf{X}, \mathcal{X}, \mathcal{H}, \mu)} = \int [f, g]_{\mathcal{H}}\, \mathrm{d}\mu$.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, then, combining the first and last examples, we get that the space $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ of centered variables in $L^2(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module when endowed with the module action and gramian defined, for all $\Phi \in \mathcal{L}_b(\mathcal{H}_0)$, and $X, Y \in \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$, by

$$\Phi \bullet X = \Phi X \quad \text{and} \quad [X, Y]_{\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})} = \mathrm{Cov}\,(X, Y) = \mathbb{E}\,[X \otimes Y]\,.$$

In the univariate case, the measure $\hat{X}$ obtained by Theorem 1.2 is valued in the space of centered $L^2(\Omega, \mathcal{F}, \mathbb{P})$ variables and is orthogonally scattered. In the functional case, we expect it to be in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Since the latter is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module, it is natural to extend the notion of c.a.o.s. measures for such spaces and to expect $\hat{X}$ to satisfy this new property. As explained earlier, the extension of orthogonality in a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module is gramian-orthogonality leading naturally to the notion of *countably additive gramian-orthogonally scattered measures* that we now introduce.

2.4 Countably additive gramian orthogonally scattered measures

This section aims at presenting the generalization of c.a.o.s. measures to normal Hilbert modules. Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module and $(\mathsf{X}, \mathcal{X})$ a measurable space. Let ν be a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$. A c.a.g.o.s. measure W on $(\mathsf{X}, \mathcal{X}, \mathcal{H})$ with intensity operator measure ν is a mapping $W : \mathcal{X} \to \mathcal{H}$ such that, for all $A, B \in \mathcal{X}$, $[W(A), W(B)]_{\mathcal{H}} = \nu(A \cap B)$. In fact, the intensity operator measure ν can be deduced from W as in the following definition.

Definition 2.8 (c.a.g.o.s. measure)**.** *Let $\mathcal{H}_0$ be a separable Hilbert space, $\mathcal{H}$ a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module and $(\mathsf{X}, \mathcal{X})$ a measurable space. We say that $W : \mathcal{X} \to \mathcal{H}$ is a* countably additive gramian-orthogonally scattered measure *(c.a.g.o.s. measure) on $(\mathsf{X}, \mathcal{X}, \mathcal{H})$ if it is a $\mathcal{H}$-valued measure on $(\mathsf{X}, \mathcal{X})$ such that for all $A, B \in \mathcal{X}$,*

$$A \cap B = \emptyset \Rightarrow [W(A), W(B)]_{\mathcal{H}} = 0 .$$

In this case, the mapping

$$\nu_W : A \mapsto [W(A), W(A)]_{\mathcal{H}}$$

is a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ called the intensity operator measure *of W and we have that, for all $A, B \in \mathcal{X}$,*

$$\nu_W(A \cap B) = [W(A), W(B)]_{\mathcal{H}} .$$

In is straightforward to see that a c.a.g.o.s. measure W is a c.a.o.s. measure with intensity measure $\|\nu_W\|_1$ which, in particular implies that, when X is a locally-compact topological space, W is regular if and only if $\|\nu_W\|_1$ is regular. By the known integration theory for c.a.o.s. measures (see Appendix C.2), it is possible to integrate scalar-valued functions of $L^2(\mathsf{X}, \mathcal{X}, \|\nu_W\|_1)$ with respect to W, but this does not make entire use of the module structure of $\mathcal{H}$ and we would like to define an integral satisfying the natural property that for all $\Phi \in \mathcal{L}_b(\mathcal{H}_0)$, $\int \Phi \mathbb{1}_A \, \mathrm{d}W = \Phi W(A)$. More generally, as explained in Remark 1.2, we want to define integrals of operator-valued functions with respect to a c.a.g.o.s. measure. By analogy to the case of c.a.o.s. measures, we therefore need to define a L^2-kind of space to integrate operator-valued functions with respect to a p.o.v.m. In the next section we present and discuss the construction of such a space and of integration of operator-valued functions with respect to a c.a.g.o.s. measure.

2.5 Square-integrable bounded-operator-valued functions with respect to a trace-class p.o.v.m.

Let $\mathcal{H}_0$ be a separable Hilbert space, $(\mathsf{X}, \mathcal{X})$ a measurable space and ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$. Let $\mathcal{G}_0$ be another separable Hilbert space and $\Phi, \Psi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, then it is easy to check that $A \mapsto \Phi \nu(A) \Psi^{\mathsf{H}}$ defines a $\mathcal{S}_1(\mathcal{H}_0)$-valued measure. By linearity, such a definition can be extended to the case where Φ, Ψ are simple functions from X to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and it is then natural to want to provide a meaning to an integral of the type $\int_{\mathsf{X}} \Phi(t) \nu(\mathrm{d}t) \Psi(t)^{\mathsf{H}}$ where $\Phi, \Psi \in \mathbb{F}(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$ or, more generally, in $\mathbb{F}_s(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0)$. Since ν has a density with respect to any measure μ dominating $\|\nu\|_1$, the construction of such integrals is very similar to the work done in [38] but is more general as discussed in Section 6.3. This approach is a natural extension of the work done in finite dimension in [33] and is followed in [20, 27].

Definition 2.9. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\mathsf{X}, \mathcal{X})$ a measurable space, ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ with density $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Let $\Phi, \Psi \in \mathbb{F}_s(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0)$, then the* *pair (Φ,Ψ) is said to be ν-integrable if $\Phi f \Psi^{\mathsf{H}} \in \mathcal{L}^1(\mathsf{X},\mathcal{X},\mathcal{S}_1(\mathcal{G}_0),\|\nu\|_1)$ and in this case we define*

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} := \int \Phi f \Psi^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 \in \mathcal{S}_1(\mathcal{G}_0) \; . \tag{2.7}$$

If (Φ,Φ) is ν-integrable we say that Φ is square ν-integrable and denote by $\mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$ the space of square ν-integrable functions.

To check that Φ is square ν-integrable, we can replace $\|\nu\|_1$ by an arbitrary dominating measure μ (often taken as Lebesgue's measure, as in [38]), as stated in the following result.

Proposition 2.7. *Let $\mathcal{H}_0,\mathcal{G}_0$ be separable Hilbert spaces, $(\mathsf{X},\mathcal{X})$ a measurable space and ν a trace-class p.o.v.m. on $(\mathsf{X},\mathcal{X},\mathcal{H}_0)$. Let μ be a finite non-negative measure on $(\mathsf{X},\mathcal{X})$ which dominates $\|\nu\|_1$ and $g=\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Let $\Phi,\Psi \in \mathbb{F}_s\left(\mathsf{X},\mathcal{X},\mathcal{H}_0,\mathcal{G}_0\right)$. Then (Φ,Ψ) is ν-integrable if and only if $\Phi g \Psi^{\mathsf{H}} \in \mathcal{L}^1(\mathsf{X},\mathcal{X},\mathcal{S}_1(\mathcal{G}_0),\mu)$, and, in this case, we have*

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int \Phi g \Psi^{\mathsf{H}} \, \mathrm{d}\mu \; . \tag{2.8}$$

Moreover, we have

$$\Phi \in \mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu) \Leftrightarrow \Phi g^{1/2} \in \mathcal{L}^2(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu) \; , \tag{2.9}$$

and, if $\Phi,\Psi \in \mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$, then (Φ,Ψ) is ν-integrable and

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int (\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \, \mathrm{d}\mu. \tag{2.10}$$

The equivalence in (2.9) says that $\mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$ is the preimage of $\mathcal{L}^2(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)$ by the mapping

$$\begin{array}{lll} \mathbb{F}_s\left(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0)\right) & \to & \mathbb{F}(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0)) \\ \Phi & \mapsto & \Phi g^{1/2} \end{array}$$

and (2.10) can be rewritten as

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \left[\Phi g^{1/2}, \Psi g^{1/2}\right]_{\mathcal{L}^2(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)}$$

where $[\cdot,\cdot]_{\mathcal{L}^2(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)}$ is the pseudo-gramian (in the sense that is satisfies all the conditions of Definition 2.4 except (ii)) defined on $\mathcal{L}^2(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)$ in Example 2.1. This pseudo-gramian becomes a gramian on $L^2(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)$ which we recall is obtained by quotienting $\mathcal{L}^2(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)$ with the μ-a.e. equality and this new space is a normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-module. This leads easily to the following proposition.

Proposition 2.8. *Let $\mathcal{H}_0,\mathcal{G}_0$ be separable Hilbert spaces, $(\mathsf{X},\mathcal{X})$ a measurable space, ν a trace-class p.o.v.m. on $(\mathsf{X},\mathcal{X},\mathcal{H}_0)$ and $f=\frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Then $\mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$ is a left $\mathcal{L}_b(\mathcal{G}_0)$-module with module action*

$$\Theta \bullet \Phi : t \mapsto \Theta\Phi(t), \quad \Theta \in \mathcal{L}_b(\mathcal{G}_0), \Phi \in \mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$$

and the relation

$$[\Phi,\Psi]_{\mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)} := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} \quad \Phi,\Psi \in \mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu) \tag{2.11}$$

is a pseudo-gramian on $\mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$ and a gramian on the quotient space

$$\mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu) := \mathscr{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu) \Big/ \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-}a.e.\right\} \cdot$$

Moreover $\left(\mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu), [\cdot,\cdot]_{\mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)}\right)$ is a normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module and, for any finite non-negative measure μ dominating $\|\nu\|_1$ with density $g=\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$,

$$\left\{\Phi \,:\, \Phi g^{1/2} = 0 \quad \mu\text{-}a.e.\right\} = \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-}a.e.\right\} \; , \tag{2.12}$$

and the mapping $\Phi \mapsto \Phi g^{1/2}$ is a gramian-isometry from $\mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\nu)$ to $L^2(\mathsf{X},\mathcal{X},\mathcal{S}_2(\mathcal{H}_0,\mathcal{G}_0),\mu)$.

In the multivariate case (i.e. when $\mathcal{H}_0$ and $\mathcal{G}_0$ have finite dimensions) the completeness of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is proven in [33]. However completeness is not guaranteed in the infinite dimensional case, see [27], where the authors refer to [25] for a counter-example. In Section 6.4, we complete this line of thoughts by providing a necessary and sufficient condition for the completeness of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ in the general case. Since the integral of operator-valued functions with respect to a c.a.g.o.s. measure is expected to be a gramian-unitary operator, it must be defined on a complete space. A first option is then to complete the space $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ by taking the equivalence classes of Cauchy sequences such that two such sequences (U_n) and (V_n) are in the same class if $\lim(U_n - V_n) = 0$. However, the completed space is very abstract and hard to describe in an intuitive way. More concretely the uncompleteness of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ comes from the fact that we restrict ourselves to $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued functions. A more concrete complete extension of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$, as noticed in [20, Section 3.4] and [27], simply consists in extending this space to include well chosen $\mathcal{O}(\mathcal{H}_0, \mathcal{G}_0)$-valued functions. We summarize their construction in the following section.

2.6 Square-integrable operator-valued functions with respect to a trace-class p.o.v.m.

Definition 2.10. *Let $\mathcal{H}_0, \mathcal{G}_0$ be two separable Hilbert spaces, ν be a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Let $\Phi, \Psi \in \mathbb{F}_{\mathcal{O}}(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0)$, then the pair (Φ, Ψ) is said to be ν-integrable if the three following assertions hold.*

(i) $\mathrm{Im}(f^{1/2}) \subset \mathcal{D}(\Phi)$ *and* $\mathrm{Im}(f^{1/2}) \subset \mathcal{D}(\Psi)$ $\|\nu\|_1$*-a.e.*

(ii) $\Phi f^{1/2}$ *and* $\Psi f^{1/2}$ *are* $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$*-valued.*

(iii) $(\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} \in \mathcal{L}^1(\mathsf{X}, \mathcal{X}, \mathcal{S}_1(\mathcal{G}_0), \|\nu\|_1)$.

In this is the case, we define for all $A \in \mathcal{X}$,

$$\int_A \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} := \int_A (\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} \, \mathrm{d}\|\nu\|_1 \in \mathcal{S}_1(\mathcal{G}_0) \ . \tag{2.13}$$

If (Φ, Φ) is ν-integrable, then Φ is said to be square ν-integrable and we denote by $\mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ the set of square ν-integrable functions.

Note that, when Φ and Ψ are $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued, we can write $(\Phi f^{1/2})(\Psi f^{1/2})^{\mathsf{H}} = \Phi f \Psi^{\mathsf{H}}$ because the adjoint of Ψ exists. In the general case the latter exists only when $\mathcal{D}(\Psi)$ is dense in $\mathcal{H}_0$. The left hand side term of (2.13) should therefore be taken only as a shorthand notation for the right hand side term which makes sense because of (ii). As previously, we can show that $\|\nu\|_1$ can be replaced by any finite non-negative measure μ dominating $\|\nu\|_1$ and the following characterization holds.

Proposition 2.9. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\mathsf{X}, \mathcal{X})$ a measurable space and ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$. Let μ be a finite non-negative measure dominating $\|\nu\|_1$ and $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$. Let $\Phi, \Psi \in \mathbb{F}_{\mathcal{O}}(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0)$, then (Φ, Ψ) is ν-integrable if and only if it satisfies*

(i') $\mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Phi)$ *and* $\mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Psi)$ μ*-a.e.*

(ii') $\Phi g^{1/2}$ *and* $\Psi g^{1/2}$ *are* $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$*-valued.*

(iii') $(\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \in \mathcal{L}^1(\mathsf{X}, \mathcal{X}, \mathcal{S}_1(\mathcal{G}_0), \mu)$.

In this case we have for all $A \in \mathcal{X}$,

$$\int_A \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int_A (\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \, \mathrm{d}\mu \ . \tag{2.14}$$

Moreover, we have

$$\Phi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \Leftrightarrow \begin{cases} \mathrm{Im}(g^{1/2}) \subset \mathcal{D}(\Phi) \ \mu\text{-a.e.} \\ \Phi g^{1/2} \in \mathcal{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu) \end{cases} \tag{2.15}$$

and, if $\Phi, \Psi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, then (Φ, Ψ) is ν-integrable and

$$\int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} = \int (\Phi g^{1/2})(\Psi g^{1/2})^{\mathsf{H}} \, \mathrm{d}\mu = \left[\Phi g^{1/2}, \Psi g^{1/2}\right]_{\mathcal{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)} . \tag{2.16}$$

Similarly as before, we get the following (stronger) result.

Theorem 2.10. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\mathsf{X}, \mathcal{X})$ a measurable space, ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Then $\mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is a $\mathcal{L}_b(\mathcal{G}_0)$-module with module action*

$$\Theta \bullet \Phi : t \mapsto \Theta\Phi(t), \quad \Theta \in \mathcal{L}_b(\mathcal{G}_0), \Phi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$$

and the relation

$$[\Phi, \Psi]_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)} := \int \Phi \mathrm{d}\nu \Psi^{\mathsf{H}} \quad \Phi, \Psi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) , \tag{2.17}$$

is a pseudo-gramian on $\mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and a gramian on the quotient space

$$\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) := \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) \Big/ \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}\right\} .$$

Moreover, $\left(\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu), [\cdot,\cdot]_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)}\right)$ is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module and, for any finite non-negative measure μ dominating $\|\nu\|_1$ with density $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then

$$\left\{\Phi \,:\, \Phi g^{1/2} = 0 \quad \mu\text{-a.e.}\right\} = \left\{\Phi \,:\, \Phi f^{1/2} = 0 \quad \|\nu\|_1\text{-a.e.}\right\} , \tag{2.18}$$

and the mapping $\Phi \mapsto \Phi g^{1/2}$ is a gramian unitary operator from $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ to $L^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$.

We now have three different kinds of L^2 spaces for operator-valued functions which are linked by the easily verified inclusions

$$L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu) \subset \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu) , \tag{2.19}$$

where the second inclusion is an isometric embedding and the first one a continuous embedding. More precisely, if $\Phi \in L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$, then

$$\|\Phi\|_{L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)} \geq \|\Phi\|_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)} = \|\Phi\|_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)} , \tag{2.20}$$

with the convention that $\|\Phi\|_{L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)} = \infty$ if $\Phi \notin L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$.

We conclude this section by the following theorem stating that $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ satisfies the usual density properties.

Theorem 2.11. *Let $\mathcal{H}_0, \mathcal{G}_0$ be two separable Hilbert spaces, $(\mathsf{X}, \mathcal{X})$ a measurable space, and ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$. Then the space $L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ is dense in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. In particular, this implies that the space* $\mathrm{Span}\,(t \mapsto \mathbb{1}_A(t)\Phi \,:\, A \in \mathcal{X}, \Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$ *of simple $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued functions is dense in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and that, if T is an l.c.a. group and ν is a regular p.o.v.m. on $(\mathrm{T}, \mathcal{B}(\mathrm{T}))$, the space* $\mathrm{Span}\left(t \mapsto \chi(t)\Phi \,:\, \chi \in \hat{\mathrm{T}}, \Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)\right)$ *of $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued trigonometric polynomials is dense in $\mathsf{L}^2(\mathrm{T}, \mathcal{B}(\mathrm{T}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$.*

2.7 Integration with respect to a c.a.g.o.s. measure

Now that we have constructed a normal Hilbert module of square-integrable operator-valued functions with respect to a trace-class p.o.v.m. we can provide a gramian-isometry which will give a meaning to integrals of operator-valued functions with respect to a c.a.g.o.s. measure. Let $(\mathsf{X}, \mathcal{X})$ be a measurable space, $\mathcal{H}_0, \mathcal{G}_0$ two separable Hilbert spaces, to simplify the construction we will consider the normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-modules $\mathcal{H} := \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and normal Hilbert $\mathcal{L}_b(\mathcal{G}_0)$-modules $\mathcal{G} := \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space. We restrict ourselves to this special case because it is the one which will be useful for spectral analysis of functional processes and this avoids technicalities necessary to define the following integrals for more general $\mathcal{H}$ and $\mathcal{G}$.

Theorem 2.12. *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $\mathcal{G} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. Let W be a c.a.g.o.s. measure on $(\mathsf{X}, \mathcal{X}, \mathcal{H})$ with intensity operator measure ν_W. Then there exists a unique gramian isometry*

$$I_W^{\mathcal{G}_0} : \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W) \to \mathcal{G}$$

such that, for all $A \in \mathcal{X}$ and $\Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$,

$$I_W^{\mathcal{G}_0}(\mathbb{1}_A \Phi) = \Phi W(A) \; .$$

Moreover, $I_W^{\mathcal{G}_0}$ is gramian-unitary from $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$ to $\overline{\mathrm{Span}}^{\mathcal{G}} (\Phi W(A) \; : \; A \in \mathcal{X}, \Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0))$.

Definition 2.11 (Integral and density with respect to a c.a.g.o.s. measure)**.** *Under the assumptions of Theorem 2.12, we use an integral sign to denote $I_W^{\mathcal{G}_0}(\Phi)$ for $\Phi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Namely, we write*

$$\int \Phi \, \mathrm{d}W = \int \Phi(t) \, W(\mathrm{d}t) := I_W^{\mathcal{G}_0}(\Phi) \; . \tag{2.21}$$

It is easy to show that, for any $\Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, the mapping

$$V : A \mapsto \int_A \Phi \, \mathrm{d}W = I_W^{\mathcal{G}_0}(\mathbb{1}_A \Phi)$$

is a c.a.g.o.s. measure on $(\mathsf{X}, \mathcal{X}, \mathcal{G})$ with intensity operator measure $\Phi \nu \Phi^{\mathsf{H}} : A \mapsto \int_A \Phi \mathrm{d}\nu \Phi^H$. We say that V has density Φ *with respect to W and write $\mathrm{d}V = \Phi \mathrm{d}W$ (or, equivalently, $V(\mathrm{d}t) = \Phi(t) W(\mathrm{d}t)$). Given $\Phi \in \mathbb{F}_{\mathcal{O}}(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0)$, we will denote*

- *by $\mathbb{M}_\Phi$ the set of trace-class p.o.v.m. ν on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ such that $\Phi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$,*
- *by $\hat{\mathcal{S}}_\Phi$ the set of c.a.g.o.s. measures on $(\mathsf{X}, \mathcal{X}, \mathcal{H})$ whose intensity operator measure is in $\mathbb{M}_\Phi$,*
- *and by $\hat{F}_\Phi$ the mapping which maps any c.a.g.o.s. measure $W \in \hat{\mathcal{S}}_\Phi$ to the c.a.g.o.s. measure with density Φ with respect to W.*

As for c.a.o.s. measures we show the converse property deriving a c.a.g.o.s. measure from a gramian isometry.

Theorem 2.13. *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. Let $\mathcal{H}_0$ be separable Hilbert spaces, $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$. Then for any gramian-isometry $w : \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0), \nu) \to \mathcal{H}$, there exists a unique c.a.g.o.s. measure W on $(\mathsf{X}, \mathcal{X}, \mathcal{H})$ with intensity operator measure ν such that for all $\Phi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0), \nu)$,*

$$w(\Phi) = \int \Phi \, \mathrm{d}W \; . \tag{2.22}$$

3 Functional weakly-stationary processes in the spectral domain

Now, we have all the tools to derive spectral analysis for functional weakly-stationary processes. We follow Section 4.2 of [20] and then study linear filtering based on the spectral representation thereby constructed.

3.1 Spectral representation of a centered weakly-stationary $\mathcal{H}_0$-valued process and definition of linear filtering

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H}_0$ a separable Hilbert space and $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Let $X = (X_t)_{t \in \mathrm{T}} \in \mathcal{H}^{\mathrm{T}}$ be a centered, weakly-stationary, $\mathcal{H}_0$-valued process indexed by an l.c.a. group $(\mathrm{T}, +)$. By analogy to the univariate case, and taking into account the module structure of $\mathcal{H}$, let us define the *modular time domain* of X as the submodule of $\mathcal{H}$ generated by the X_t's, that is

$$\mathcal{H}^X := \overline{\operatorname{Span}}^{\mathcal{H}} \left(\Phi X_t \ : \ \Phi \in \mathcal{L}_b(\mathcal{H}_0),\, t \in \mathrm{T}\right) .$$

Similarly, given another separable Hilbert space $\mathcal{G}_0$, we define

$$\mathcal{H}^{X,\mathcal{G}_0} := \overline{\operatorname{Span}}^{\mathcal{G}} \left(\Phi X_t \ : \ \Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0),\, t \in \mathrm{T}\right)$$

which is a submodule of $\mathcal{G} := \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. For all $h \in \mathrm{T}$, define (using Proposition 2.5) the shift operator of lag h as the unique gramian-unitary operator $U_h^X : \mathcal{H}^X \to \mathcal{H}^X$ which maps X_t to X_{t+h} for all $t \in \mathrm{T}$. As in the univariate case (see Lemma 1.1), weak stationarity is characterized by the representation properties of U^X seen in Definition 2.7.

Lemma 3.1. *Let $X = (X_t)_{t \in \mathrm{T}}$ be a centered L^2 $\mathcal{H}_0$-valued process. Then X is weakly stationary if and only if U^X is a c.g.u.r. of T on $\mathcal{H}^X$.*

In particular (see also Remark 1.3), continuity of U^X is equivalent to weak-continuity of Γ_X from Definition 1.3 (see [20, Proposition 2.5.2]). The following theorem (see [20, Theorem 4.2.2, Theorem 4.2.4]) gives **R1**, **R2**.

Theorem 3.2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathrm{T}, +)$ an l.c.a. group. Let $\mathcal{H}_0$ be a separable Hilbert space and set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Let $X := (X_t)_{t \in \mathrm{T}} \in \mathcal{H}^{\mathrm{T}}$. Then X is weakly stationary if and only if there exists a regular c.a.g.o.s. measure $\hat{X}$ on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{H})$ such that*

$$X_t = \int_{\hat{\mathrm{T}}} \chi(t)\ \hat{X}(\mathrm{d}\chi) \quad \textit{for all} \quad t \in \mathrm{T} . \tag{3.1}$$

In this case, $\hat{X}$ is uniquely determined by (3.1) *and is called the* spectral representation *of X. The intensity operator measure ν_X of $\hat{X}$ is called the* spectral operator measure *of X. It is a regular trace-class p.o.v.m. on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{H}_0)$ and is the unique regular p.o.v.m. satisfying*

$$\Gamma_X(h) = \int_{\hat{\mathrm{T}}} \chi(t)\, \nu_X(\mathrm{d}\chi) \quad \textit{for all} \quad h \in \mathrm{T} . \tag{3.2}$$

Note that, like for Bochner's theorem in the univariate case, Relation (3.2) can be obtained without using stochastic processes and this result can also be used to derive spectral analysis for weakly-stationary stochastic processes. This will be discussed in Section 6. With these results, we can now define linear filtering for functional weakly stationary processes in the spectral domain. First, we characterize integration with respect to $\hat{X}$ by the following result.

Corollary 3.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathrm{T}, +)$ an l.c.a. group. Let $\mathcal{H}_0$ be a separable Hilbert space and set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Let $X := (X_t)_{t \in \mathrm{T}}$ be a centered, weakly stationary, $\mathcal{H}_0$-valued process with spectral representation $\hat{X}$ and spectral operator measure ν_X. Then for any separable Hilbert space $\mathcal{G}_0$, the mapping $\Psi \mapsto \int \Psi \, \mathrm{d}\hat{X}$ is the unique gramian isometry from $\mathsf{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)$ to $\mathcal{G} = \mathcal{M}(\Omega, \mathcal{G}, \mathcal{G}_0, \mathbb{P})$ such that*

$$\int_{\hat{\mathrm{T}}} \chi(t) \Phi\ \hat{X}(\mathrm{d}\chi) = \Phi X_t \quad \textit{for all} \quad t \in \mathrm{T} \quad \textit{and} \quad \Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0) .$$

This mapping is, in addition, gramian-unitary from $\mathsf{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)$ to $\mathcal{H}^{X,\mathcal{G}_0}$. In particular, we have that $\overline{\operatorname{Span}}^{\mathcal{G}} \left(\Phi \hat{X}(A) \,:\, A \in \mathcal{B}(\hat{\mathrm{T}}), \Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0) \right) = \mathcal{H}^{X,\mathcal{G}_0}$.

Denote by $\mathfrak{F}_{\mathcal{H}_0}$ the function which maps a centered weakly stationary $\mathcal{H}_0$-valued process X indexed by T to its spectral representation $\hat{X}$ valued in the space of regular c.a.g.o.s. measures on $(\hat{\mathrm{T}}, \mathcal{B}(\mathrm{T}), \mathcal{H})$. Then, by Theorem 3.2, $\mathfrak{F}_{\mathcal{H}_0}$ is bijective and we can define linear filtering using the notions introduced in Definition 2.11 for c.a.g.o.s. measures.

Definition 3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\mathcal{H}_0, \mathcal{G}_0$ separable Hilbert spaces and $(\mathrm{T}, +)$ an l.c.a. group. Call $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $\mathcal{G} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. For $\Phi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{H}_0, \mathcal{G}_0\right)$ we denote by $\mathcal{S}_\Phi$ the set of centered weakly-stationary $\mathcal{H}_0$-valued processes indexed by T whose spectral operator measure is in $\mathbb{M}_\Phi$ (or equivalently $\mathcal{S}_\Phi = \mathfrak{F}_{\mathcal{H}_0}^{-1}\left(\hat{\mathcal{S}}_\Phi\right)$). We define the filter F_Φ with transfer operator function Φ as the operator defined on $\mathcal{S}_\Phi$ by $F_\Phi = \mathfrak{F}_{\mathcal{G}_0}^{-1} \circ \hat{F}_\Phi \circ (\mathfrak{F}_{\mathcal{H}_0})$. In other words, for all $X \in \mathcal{S}_\Phi$, $Y = F_\Phi(X)$ is the $\mathcal{G}_0$-valued weakly stationary process satisfying $\mathrm{d}\hat{Y} = \Phi \mathrm{d}\hat{X}$, that is, for all $t \in \mathrm{T}$,*

$$Y_t = \int_{\hat{\mathrm{T}}} \chi(t) \Phi(\chi)\, \hat{X}(\mathrm{d}\chi) . \tag{3.3}$$

In order to compose filters defined this way, we now need to explain how to compose mappings of the form $\hat{F}_\Phi$, which is closely linked to composition of square-integrable operator-valued functions with respect to trace-class p.o.v.m. In the next section we explain how this is done and conclude giving results on composition and inversion of filters.

3.2 Composition and inversion of linear filters

Let $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ be separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$. Let $\Phi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, then $\Phi\nu\Phi^{\mathsf{H}} : A \mapsto \int_A \Phi \mathrm{d}\nu\Phi^{\mathsf{H}}$ is a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{G}_0)$ and the space $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ is characterized by the following theorem.

Theorem 3.4. *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ separable Hilbert spaces and ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$. Let $\Phi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and $\Psi \in \mathbb{F}_{\mathcal{O}}(\mathsf{X}, \mathcal{X}, \mathcal{G}_0, \mathcal{I}_0)$. Then*

$$\Psi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}}) \Leftrightarrow \Psi\Phi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu) . \tag{3.4}$$

Moreover, the following assertions hold.

(a) For all $\Psi, \Theta \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$,

$$(\Psi\Phi)\nu(\Theta\Phi)^{\mathsf{H}} = \Psi(\Phi\nu\Phi^{\mathsf{H}})\Theta^{\mathsf{H}} .$$

(b) The mapping $\Psi \mapsto \Psi\Phi$ is a well defined gramian-isometry from $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ to $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu)$.

(c) Suppose moreover that Φ is injective $\|\nu\|_1$-a.e., then we have that

$$\Phi^{-1} \in \mathscr{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \Phi\nu\Phi^{\mathsf{H}}) ,$$

where we define $\Phi^{-1}(\lambda) := \left(\Phi(\lambda)_{|\mathcal{D}(\Phi(\lambda)) \to \mathrm{Im}(\Phi(\lambda))}\right)^{-1}$ with domain $\mathrm{Im}(\Phi(\lambda))$ for all $\lambda \in \{\Phi \text{ is injective}\}$ and $\Phi^{-1}(\lambda) = 0$ otherwise.

The following corollaries are obtained from this theorem and allow us to deal with the composition and inversion of filters.

Corollary 3.5. *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces and $\Phi \in \mathbb{F}_{\mathcal{O}}(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0)$. Let $W \in \hat{\mathcal{S}}_\Phi$ and $V = \hat{F}_\Phi(W)$, then for all separable Hilbert space $\mathcal{I}_0$, denoting by $\mathcal{I} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{I}_0, \mathbb{P})$, we have*

$$\overline{\mathrm{Span}}^{\mathcal{I}}(\Psi V(A) \,:\, A \in \mathcal{X}, \Psi \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)) \lesssim \overline{\mathrm{Span}}^{\mathcal{I}}(\Psi W(A) \,:\, A \in \mathcal{X}, \Psi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{I}_0)) . \tag{3.5}$$

Corollary 3.6 (Composition of filters). *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space, $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ separable Hilbert spaces and $\Phi \in \mathbb{F}_{\mathcal{O}}(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0)$, $\Psi \in \mathbb{F}_{\mathcal{O}}(\mathsf{X}, \mathcal{X}, \mathcal{G}_0, \mathcal{I}_0)$. Let $W \in \hat{\mathcal{S}}_\Phi$, then $W \in \hat{\mathcal{S}}_{\Psi\Phi}$ if and only if $\hat{F}_\Phi(W) \in \hat{\mathcal{S}}_\Psi$ and in this case*

$$\hat{F}_\Psi \circ \hat{F}_\Phi(W) = \hat{F}_{\Psi\Phi}(W). \tag{3.6}$$

Remark 3.1. *The identity (3.6) can be reformulated with integrals as follows. If $\mathrm{d}V = \Phi \mathrm{d}W$ (i.e. $V = \hat{F}_\Phi(W)$) then for all $\Psi \in \mathcal{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu_V)$ and $A \in \mathcal{X}$,*

$$\int_A \Psi \,\mathrm{d}V = \int_A \Psi\Phi \,\mathrm{d}W. \tag{3.7}$$

Corollary 3.7 (Inversion of filters). *Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces and set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $\mathcal{G} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{G}_0, \mathbb{P})$. Let W be a c.a.g.o.s. measure on $(\mathsf{X}, \mathcal{X}, \mathcal{H})$ with intensity operator measure ν_W and $\Phi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)$. Suppose that Φ is injective $\|\nu_W\|_1$-a.e. and let $V = F_\Phi(W)$. Then $W = F_{\Phi^{-1}}(V)$ and the $\lesssim$ in Equation (3.5) becomes a $\cong$.*

Remark 3.2. *These corollaries have time domain counterparts. Namely, if X is a centered weakly stationary $\mathcal{H}_0$-valued process indexed by an l.c.a. group $(\mathrm{T}, +)$ and $\Phi \in \mathscr{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)$. Then, calling $Y = F_\Phi(X)$, we get that $\mathcal{H}^{Y,\mathcal{I}_0} \lesssim \mathcal{H}^{X,\mathcal{I}_0}$ for all separable Hilbert space $\mathcal{I}_0$. Moreover, if $\Psi \in \mathbb{F}_{\mathcal{O}}\left(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{G}_0, \mathcal{I}_0\right)$, then $X \in \mathcal{S}_{\Psi\Phi}$ if and only if $Y \in \mathcal{S}_\Psi$ and in this case $F_\Psi(Y) = F_{\Psi\Phi}(X)$. Finally, if Φ is injective $\|\nu_X\|_1$-a.e., then $X = F_{\Phi^{-1}}(Y)$ and $\mathcal{H}^{Y,\mathcal{I}_0} \cong \mathcal{H}^{X,\mathcal{I}_0}$.*

4 Applications to functional time series

In the following applications we consider discrete time processes, that is $\mathrm{T} = \mathbb{Z}$ and $\hat{\mathrm{T}} = \mathbb{T}$, valued in a separable Hilbert space $\mathcal{H}_0$.

4.1 Functional ARMA processes

Let p be a positive integer and consider the p-order linear recursive equation

$$Y_t = \sum_{k=1}^{p} A_k Y_{t-k} + \epsilon_t \; , \quad t \in \mathbb{Z} \; , \tag{4.1}$$

where $\epsilon = (\epsilon_t)_{t\in\mathbb{Z}}$ is a centered weakly stationary process valued in $\mathcal{H}_0$ and $A_1, \dots, A_p \in \mathcal{L}_b(\mathcal{H}_0)$. If ϵ is a white noise (that is, it admits a constant spectral density operator), then Equation (4.1) is called a (functional) auto-regressive process of order p (AR(p)) equation. If ϵ can be written for some positive integer q as

$$\epsilon_t = Z_t + \sum_{k=1}^{q} B_k Z_{t-k} \; , \quad t \in \mathbb{Z} \; ,$$

where $Z = (Z_t)_{t\in\mathbb{Z}}$ is a centered white noise valued in $\mathcal{H}_0$ and $B_1, \dots, B_p \in \mathcal{L}_b(\mathcal{H}_0)$, then ϵ is called a (functional) moving average process of order q (MA(q)) and Eq. (4.1) is called a (functional) auto-regressive moving average process of order (p, q) (ARMA(p, q)) equation. Weakly stationary solutions of AR(p) or ARMA(p, q) equations are called AR(p) or ARMA(p, q) processes.

The existence of a weakly stationary solution to Eq. (4.1) occurs under the assumption that

$$Q(z) = \mathrm{Id}_{\mathcal{H}_0} - \sum_{k=1}^{p} A_k z^k \quad \text{is invertible in } \mathcal{L}_b(\mathcal{H}_0) \text{ for all } z \in \mathbb{C} \text{ such that } |z| = 1. \tag{4.2}$$

It is usually proven by using an explicit expansion of the form (see [36, Corollary 2.2] for the Banach space valued case and the references in the proof)

$$Y_t = \sum_{k\in\mathbb{Z}} \psi_k \epsilon_{t-k} \; , \quad t \in \mathbb{Z} \; , \tag{4.3}$$

where $(\psi_k)_{k\in\mathbb{Z}} \in \mathcal{L}_b(\mathcal{H}_0)$ and the series $\sum_{k\in\mathbb{Z}} \psi_k$ converges absolutely in $\mathcal{L}_b(\mathcal{H}_0)$.

Now, note that (4.2) implies that $\Phi^{-1} \in \mathbb{F}_b(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0)))$ with $\Phi(\lambda) = Q(\mathrm{e}^{-\mathrm{i}\lambda})$ for all $\lambda \in \mathbb{R}$. Thus, Corollary 3.6 immediately gives that $Y = F_{\Phi^{-1}}(\epsilon)$ is a solution of (4.1) and Corollary 3.7 that it is the unique one which is weakly stationary. Then the representation (4.3) holds as an immediate consequence of the fact that $z \mapsto Q^{-1}(z)$ is homomorhic on a ring containing the unit circle, so that

$$\Phi^{-1}(\lambda) = Q^{-1}(\mathrm{e}^{-\mathrm{i}\lambda}) = \sum_{k\in\mathbb{Z}} \psi_k \mathrm{e}^{-\mathrm{i}\lambda k} \; ,$$

where $(\psi_k)_{k\in\mathbb{Z}}$ has exponential decay at $\pm\infty$.

4.2 Functional long-memory processes

Processes with long-memory have a non-summable autocovariance function and therefore do not satisfy the assumptions of [31, 30, 38]. However, in the univariate case, spectral analysis is widely used for such processes and the goal of this section is to show how the more general spectral theory presented in Section 3 can by used to define long-memory for processes valued in a separable Hilbert space $\mathcal{H}_0$. Results on long-memory $\mathcal{H}_0$-valued processes have been derived recently using a time domain definition, namely $X_t = \sum_{k=0}^{+\infty} (k+1)^{-N} \epsilon_{t-k}$ where $(\epsilon_t)_{t\in\mathbb{Z}}$ is a white noise in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $N \in \mathcal{L}_b(\mathcal{H}_0)$ is a normal operator. We refer to [14] and the references therein for details. In particular, existence of the process $X = (X_t)_{t\in\mathbb{Z}}$ (i.e. the L^2 convergence of the series) is shown, under assumptions on N

and $(\epsilon_t)_{t\in\mathbb{Z}}$, in [14, Lemma A.1]. In this section, we define, under the same assumptions, a long-memory $\mathcal{H}_0$-valued process from its spectral representation by naturally extending the celebrated (univariate) autoregressive fractionally integrated moving average (ARFIMA) models, and explain how it is related to the process X defined above. For later use, we recall that (see [9, Theorem 9.4.6, Proposition 9.4.7]), if $N \in \mathcal{L}_b(\mathcal{H}_0)$ is normal, then there exists a σ-finite measure space $(\mathsf{X}, \mathcal{X}, \mu)$ and a function $d \in L^\infty(\mathsf{X}, \mathcal{X}, \mu)$, such that N has a singular values decomposition of the type $UNU^{\mathsf{H}} = M_d$ where $U : \mathcal{H}_0 \to L^2(\mathsf{X}, \mathcal{X}, \mu)$ is unitary and M_d is the multiplicative operator on $L^2(\mathsf{X}, \mathcal{X}, \mu)$ associated to d, that is $M_d : f \mapsto (s \mapsto d(s)f(s))$. In the following, we will also denote the open and closed unit discs by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$ and we use the notation $(1-z)^A$ for some $A \in \mathcal{L}_b(\mathcal{H}_0)$ and $z \in \mathbb{D} \setminus \{1\}$. This must be understood as

$$(1-z)^A = \exp(A \ln(1-z)) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(A \ln(1-z)\right)^k ,$$

where ln denotes the principal complex logarithm, so that $z \mapsto \ln(1-z)$ is holomorphic on $\mathbb{C} \setminus [1, \infty)$. Finally, if $Y : \Omega \to L^2(\mathsf{X}, \mathcal{X}, \mu)$ is a random variable, we make the implicit assumption that $(\omega, s) \mapsto Y(\omega)(s)$ is measurable from $(\Omega \times \mathsf{X}, \mathcal{F} \otimes \mathcal{X})$ to $(\mathbb{C}, \mathcal{B}(\mathbb{C}))$ and we define for all $s \in \mathsf{X}$, $Y(s) : \omega \mapsto Y(\omega)(s)$.

Proposition 4.1. *Let $\mathcal{H}_0$ be a separable Hilbert space, $N \in \mathcal{L}_b(\mathcal{H}_0)$ be a normal operator with singular values decomposition $UNU^{\mathsf{H}} = M_d$ as above. Let $h : s \mapsto \Re(d(s))$ and*

$$\Phi : \begin{array}{ccc} \mathbb{T} & \to & \mathcal{L}_b(\mathcal{H}_0) \\ \lambda & \mapsto & \left(1 - \mathrm{e}^{-\mathrm{i}\lambda}\right)^{N - \mathrm{Id}} \end{array} .$$

Let $\epsilon := (\epsilon_t)_{t\in\mathbb{Z}}$ be a white noise in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and $\sigma_U^2 : s \mapsto \mathbb{E}\left[|(U\epsilon_0)(s)|^2\right]$. Suppose that

$$h > \frac{1}{2} \ \mu\text{-a.e.} \quad \text{and} \quad \int_{\mathsf{X}} \frac{\sigma_U^2(s)}{2h(s) - 1} \, \mu(\mathrm{d}s) < +\infty . \tag{4.4}$$

Then $\epsilon \in \mathcal{S}_\Phi$ and there exists $C \in \mathcal{L}_b(\mathcal{H}_0)$ and $(\Delta_k)_{k\in\mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0)^{\mathbb{N}}$ with $\sum_{k\geq 0} \|\Delta_k\| < +\infty$ such that for all $t \in \mathbb{Z}$,

$$[F_\Phi(\epsilon)]_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t} \left(1 - \mathrm{e}^{-\mathrm{i}\lambda}\right)^{N - \mathrm{Id}} \hat{\epsilon}(\mathrm{d}\lambda) = C \left(\sum_{k=0}^{\infty} (k+1)^{-N} \epsilon_{t-k}\right) + \sum_{k=0}^{\infty} \Delta_k \epsilon_{t-k} , \tag{4.5}$$

where, in the right-hand-side, the first series converges in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ and the second series is absolutely convergent in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$.

Adopting the univariate definition, we call the process $F_\Phi(\epsilon)$ in (4.5) a functional $\mathrm{ARFIMA}(0, \mathrm{Id} - N, 0)$ process, the functional $\mathrm{ARFIMA}(p, \mathrm{Id} - N, q)$ process corresponding to the case where ϵ is a functional $\mathrm{ARMA}(p, q)$ process. Thus, the second equality in (4.5) says that, up to the application of the operator C and to the additional short-memory process $\sum_{k=0}^{\infty} \Delta_k \epsilon_{t-k}$, $t \in \mathbb{Z}$, the process $X_t = \sum_{k=0}^{\infty} (k+1)^{-N} \epsilon_{t-k}$, $t \in \mathbb{Z}$, coincide with a functional $\mathrm{ARFIMA}(0, \mathrm{Id} - N, 0)$ process.

The proof of Proposition 4.1 relies on three lemmas.

Lemma 4.2. *For all $z \in \mathbb{C}$ with $\Re(z) > -\frac{1}{2}$, we have*

$$\int_{-\pi}^{\pi} \left|(1 - \mathrm{e}^{-\mathrm{i}\lambda})^z\right|^2 \mathrm{d}\lambda \leq \frac{\pi^{2(\Re(z)+1)}}{2\Re(z) + 1} \mathrm{e}^{\pi|\Im(z)|} .$$

Proof. Let $z \in \mathbb{C}$ with $\Re(z) > -\frac{1}{2}$, then it can be shown that, for all $\lambda \in (-\pi, \pi] \setminus \{0\}$, $\left|(1 - \mathrm{e}^{-\mathrm{i}\lambda})^z\right|^2 = \left|1 - \mathrm{e}^{-\mathrm{i}\lambda}\right|^{2\Re(z)} \mathrm{e}^{-2\Im(z) b(\mathrm{e}^{-\mathrm{i}\lambda})}$, where $b(\mathrm{e}^{-\mathrm{i}\lambda})$ denotes the argument of $1 - \mathrm{e}^{-\mathrm{i}\lambda}$ that belongs to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. It follows that $\mathrm{e}^{-2\Im(z) b(\mathrm{e}^{-\mathrm{i}\lambda})} \leq \mathrm{e}^{\pi|\Im(z)|}$. Using that $\frac{|\lambda|}{\pi} \leq |\sin(\lambda/2)| \leq \frac{|\lambda|}{2}$ for all $\lambda \in (-\pi, \pi)$ and $\Re(z) > -1/2$, we easily get

$$\int_{-\pi}^{\pi} \left|1 - \mathrm{e}^{-\mathrm{i}\lambda}\right|^{2\Re(z)} \mathrm{d}\lambda = \int_{-\pi}^{\pi} |2\sin(\lambda/2)|^{2\Re(z)} \, \mathrm{d}\lambda \leq \pi \int_0^{\pi} |\lambda|^{2\Re(z)} \, \mathrm{d}\lambda = \frac{\pi^{2(\Re(z)+1)}}{2\Re(z) + 1} .$$

The result follows. □

Lemma 4.3. *Let E be a Banach space and $(a_n)_{n\in\mathbb{N}} \in E^{\mathbb{N}}$ such that $\|a_n\|_E \xrightarrow[n\to+\infty]{} 0$ and the series $\sum \|a_n - a_{n+1}\|_E$ converges. Then for all $z_0 \in \mathbb{D}\setminus\{1\}$, the series $\sum_{n=0}^{\infty} a_n z_0^n$ converges in E and the mapping $z \mapsto \sum_{n=0}^{\infty} a_n z^n$ is uniformly continuous on $[0, z_0]$.*

Proof. By assumption on (a_n), $\sum a_n z^n$ is a power series valued in E with convergence radius at least equal to 1, hence is uniformly continuous on the open disk with radius 1. When $|z_0| = 1$, the result follows using Abel's transform. □

Lemma 4.4. *Let $\mathcal{H}_0$ be a separable Hilbert space, $N \in \mathcal{L}_b(\mathcal{H}_0)$ be a normal operator with singular values decomposition $UNU^{\mathsf{H}} = M_d$ on $L^2(\mathsf{X}, \mathcal{X}, \mu)$ as above. Define*

$$\varrho = \mu\text{-}\operatorname*{essinf}_{s\in\mathsf{X}} \Re(d(s)) \;.$$

Then there exist $C \in \mathcal{L}_b(\mathcal{H}_0)$ and $(\Delta_k)_{k\in\mathbb{N}} \in \mathcal{L}_b(\mathcal{H}_0)^{\mathbb{N}}$ with $\|\Delta_k\| = O\left(k^{-1-\varrho}\right)$ such that, for all $z \in \mathbb{D}$,

$$(1-z)^{N-\mathrm{Id}} = C\left(\sum_{k=0}^{\infty}(k+1)^{-N} z^k\right) + \sum_{k=0}^{\infty} \Delta_k z^k \;, \tag{4.6}$$

where the series on the right-hand side are $\mathcal{L}_b(\mathcal{H}_0)$-valued power series with convergence radius at least equal to 1. Moreover, if $\varrho > 0$, then the identity (4.6) holds for all $z \in \overline{\mathbb{D}}\setminus\{1\}$, where the two series of the right-hand side still converge in $\mathcal{L}_b(\mathcal{H}_0)$.

Proof. The proof is three steps. We first show Relation (4.6) for all $z \in \mathbb{D}$, then that $\|\Delta_k\| = O\left(k^{-1-\varrho}\right)$ and finally extend the relation to $z \in \overline{\mathbb{D}}\setminus\{1\}$ when $\varrho > 0$.

Step 1. Let $z \in \mathbb{D}$, then

$$(1-z)^{N-\mathrm{Id}} = \mathrm{Id} + \sum_{k\geq 1} N_k z^k \;,$$

where for all $k \geq 1$, $N_k = \prod_{j=1}^{k}\left(\mathrm{Id} - \frac{N}{j}\right)$. Let $k_0 \geq 1$, such that $\frac{\|N\|}{k_0} < 1$ and take $k \geq k_0$, then

$$\mathrm{Id} - \frac{N}{k} = \exp\left(\ln\left(\mathrm{Id} - \frac{N}{k}\right)\right) = \exp\left(-\sum_{j\geq 1}\frac{N^j}{k^j j}\right) \;,$$

and therefore,

$$N_k = \prod_{j=1}^{k_0-1}\left(\mathrm{Id} - \frac{N}{j}\right)\exp\left(-\sum_{j\geq 1}\frac{N^j}{j}\sum_{t=k_0}^{k}\frac{1}{t^j}\right) \;.$$

Moreover, we have the following asymptotic expansions,

$$\sum_{t=k_0}^{k}\frac{1}{t} = \sum_{t=1}^{k}\frac{1}{t} - \sum_{t=1}^{k_0-1}\frac{1}{t} = \ln(k+1) + \gamma_e - \sum_{t=1}^{k_0-1}\frac{1}{t} + \frac{\alpha_k}{k}$$

and for all $j \geq 2$,

$$\sum_{t=k_0}^{k}\frac{1}{t^j} = \sum_{k=k_0}^{+\infty}\frac{1}{t^j} - \sum_{k=k+1}^{+\infty}\frac{1}{t^j} = \frac{\beta_j}{k_0^j} + \frac{\eta_{k,j}}{(j-1)k^{j-1}}$$

where γ_e is Euler's constant and $(\alpha_k)_{k\geq 1}$, $(\eta_{k,j})_{k\geq 1, j\geq 2}$ such that $\sup_{k\geq 1}|\alpha_k| < +\infty$ and $\sup_{k\geq 1, j\geq 2}|\eta_{k,j}| < +\infty$ and $\beta_j = \sum_{t=k_0}^{+\infty}\left(\frac{k_0}{t}\right)^j$ satisfies $\sup_{j\geq 2}\beta_j < +\infty$. This gives, for all $k \geq k_0$,

$$N_k = C(k+1)^{-N}\exp\left(-N\frac{\alpha_k}{k} - \sum_{j\geq 2}\frac{N^j\eta_{k,j}}{(j-1)k^{j-1}}\right)$$

where

$$C = \prod_{j=1}^{k_0-1}\left(\mathrm{Id} - \frac{N}{j}\right)\exp\left(-N\left(\gamma_e - \sum_{t=1}^{k_0-1}\frac{1}{t}\right)\right)\exp\left(-\sum_{j\geq 2}\left(\frac{N}{k_0}\right)^j\frac{\beta_j}{j}\right) \;.$$

Combining everything, we get

$$
\begin{aligned}
(1-z)^{N-\mathrm{Id}} = \mathrm{Id} &+ \sum_{k=1}^{k_0-1} \prod_{j=1}^{k} \left(\mathrm{Id} - \frac{N}{j}\right) z^k \\
&+ C \sum_{k\geq k_0} (k+1)^{-N} \exp\left(-N\frac{\alpha_k}{k} - \sum_{j\geq 2} \frac{N^j \eta_{k,j}}{(j-1)k^{j-1}}\right) z^k
\end{aligned}
$$

which leads to Relation (4.6) with

$$
\begin{aligned}
&\Delta_0 = \mathrm{Id} - C\,, \\
&\Delta_k = \prod_{j=1}^{k}\left(\mathrm{Id} - \frac{N}{j}\right) - C(k+1)^{-N}\,, && 1 \leq k \leq k_0 - 1, \\
&\Delta_k = C(k+1)^{-N}\left[\exp\left(-N\frac{\alpha_k}{k} - \sum_{j\geq 2}\frac{N^j\eta_{k,j}}{(j-1)k^{j-1}}\right) - \mathrm{Id}\right]\,, && k \geq k_0.
\end{aligned}
$$

Step 2. For all $k \geq k_0$, denoting by $\Phi_k := -N\frac{\alpha_k}{k} - \sum_{j\geq 2}\frac{N^j\eta_{k,j}}{(j-1)k^{j-1}}$, we get

$$
\|\Delta_k\| = \left\|C(k+1)^{-N}\left(\mathrm{e}^{\Phi_k} - \mathrm{Id}\right)\right\| \leq \|C\|\left\|(k+1)^{-N}\right\|\sum_{t\geq 1}\frac{\|\Phi_k\|^t}{t!} = O\left(k^{-1-\varrho}\right)
$$

because

$$
\|\Phi_k\| \leq \|N\|\frac{|\alpha_k|}{k} + \sum_{j\geq 2}\frac{\|N\|^j\eta_{k,j}}{(j-1)k^{j-1}} = \|N\|\left(\frac{|\alpha_k|}{k} + \sum_{j\geq 1}\frac{\|N\|^j}{jk^j}\eta_{k,j+1}\right) = O\left(k^{-1}\right)\,,
$$

and

$$
\left\|(k+1)^{-N}\right\| = \left\|(k+1)^{-M_d}\right\| = \left\|M_{(k+1)^{-d}}\right\| = \operatorname*{essup}_{s\in\mathsf{X}}\left|(k+1)^{-d(s)}\right| = (k+1)^{-\varrho}\,.
$$

Step 3. We now assume $\varrho > 0$ and extend (4.6) to $\overline{\mathbb{D}}\backslash\{1\}$. We already have for all $\lambda \in \mathbb{T}\backslash\{0\}$, for all $0 < a < 1$,

$$
(1 - a\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} = C\sum_{k\geq 0}(k+1)^{-N}a^k\mathrm{e}^{-\mathrm{i}\lambda k} + \sum_{k\geq 0}\Delta_k a^k \mathrm{e}^{-\mathrm{i}\lambda k}\,.
$$

Moreover, $(1-\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} = \lim_{a\to 1^-}(1-a\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}}$ by continuity of $z \mapsto (1-z)^{N-\mathrm{Id}}$ in $\overline{\mathbb{D}}\backslash\{1\}$ and $\sum_{k\geq 0}\Delta_k\mathrm{e}^{-\mathrm{i}\lambda k} = \lim_{a\to 1^-}\sum_{k\geq 0}\Delta_k a^k\mathrm{e}^{-\mathrm{i}\lambda k}$ because $\sum_{k\geq 0}\|\Delta_k\| < +\infty$. It remains to show that $\lim_{a\mapsto 1^-}\sum_{k\geq 0}(k+1)^{-N}a^k\mathrm{e}^{-\mathrm{i}\lambda k}$ exists. For all $k \in \mathbb{N}$, we have $\left\|(k+1)^{-N}\right\| = (k+1)^{-\varrho}$ and

$$
\begin{aligned}
\left\|(k+1)^{-N} - (k+2)^{-N}\right\| &= \operatorname*{essup}_{s\in\mathsf{X}}\left|(k+1)^{-d(s)} - (k+2)^{-d(s)}\right| \\
&\leq \operatorname*{essup}_{s\in\mathsf{X}}\left|(k+1)^{-d(s)}\right| \operatorname*{essup}_{s\in\mathsf{X}}\left|1 - \left(1 - \frac{1}{k+2}\right)^{d(s)}\right| \\
&= (k+1)^{-\varrho}\operatorname*{essup}_{s\in\mathsf{X}}\left|\sum_{j\geq 1}\frac{d(s)\cdots(d(s)-j+1)}{(k+2)^j}\right| \\
&\leq (k+1)^{-\varrho}\sum_{j\geq 1}\frac{\|d\|_\infty\cdots(\|d\|_\infty - j + 1)}{(k+2)^j} \\
&= (k+1)^{-\varrho}O\left(k^{-1}\right) \\
&= O\left(k^{-1-\varrho}\right)
\end{aligned}
$$

and therefore the assumptions of Lemma 4.3 are verified and Step 3 is completed. $\square$

We can now prove Proposition 4.1.

Proof of Proposition 4.1. Suppose (4.4) holds and let Γ_ϵ, ν_ϵ be the autocovariance operator function and spectral operator measure of ϵ. We successively prove that $\epsilon \in \mathcal{S}_\Phi$ (Step 1) and that the second inequality in (4.5) holds (Step 2) **Step 1.** By definition, $\epsilon \in \mathcal{S}_\Phi$ if and only if $\Phi \in \mathsf{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{L}_b(\mathcal{H}_0), \nu_\epsilon)$ which is equivalent, by Proposition 2.9 and since $\nu_\epsilon(\mathrm{d}\lambda) = \frac{\Gamma_\epsilon(0)}{2\pi}\mathrm{d}\lambda$, to

$$\int_{\mathbb{T}} \left\| (1-\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} \Gamma_\epsilon(0) \left((1-\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} \right)^{\mathsf{H}} \right\|_1 \mathrm{d}\lambda < +\infty \; . \tag{4.7}$$

This integral can be computed as follows.

$$\begin{aligned}
&\int_{\mathbb{T}} \left\| (1-\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} \Gamma_\epsilon(0) \left((1-\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} \right)^{\mathsf{H}} \right\|_1 \mathrm{d}\lambda \\
&\quad = \int_{\mathbb{T}} \left\| M_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{d-1}} U \Gamma_\epsilon(0) U^{\mathsf{H}} M^{\mathsf{H}}_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{d-1}} \right\|_1 \mathrm{d}\lambda \\
&\quad = \int_{\mathbb{T}} \left\| \mathrm{Cov}\left(M_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{d-1}} U \epsilon_0 \right) \right\|_1 \mathrm{d}\lambda \\
&\quad = \int_{\mathbb{T}} \mathbb{E}\left[\left\| M_{(1-\mathrm{e}^{-\mathrm{i}\lambda})^{d-1}} U \epsilon_0 \right\|^2_{L^2(\mathsf{X}, \mathcal{X}, \mu)} \right] \mathrm{d}\lambda \\
&\quad = \int_{\mathbb{T}} \mathbb{E}\left[\int_{\mathsf{X}} \left| (1-\mathrm{e}^{-\mathrm{i}\lambda})^{d(s)-1} \right|^2 \left| (U\epsilon_0)(s) \right|^2 \mu(\mathrm{d}s) \right] \mathrm{d}\lambda \\
&\quad = \int_{\mathsf{X}} \sigma_U^2(s) \int_{\mathbb{T}} \left| (1-\mathrm{e}^{-\mathrm{i}\lambda})^{d(s)-1} \right|^2 \mathrm{d}\lambda \, \mu(\mathrm{d}s) \; .
\end{aligned}$$

Applying Lemma 4.2 to $z = d(s) - 1$, with $\Re(z) = h(s) - 1 > -\frac{1}{2}$, gives for μ-almost every $s \in \mathsf{X}$,

$$\int_{\mathbb{T}} \left| (1-\mathrm{e}^{-\mathrm{i}\lambda})^{d(s)-1} \right|^2 \mathrm{d}\lambda \le \frac{\pi^{2h(s)}}{2h(s)-1} \mathrm{e}^{\pi|\Im(d(s))-1|} \le \frac{K}{2h(s)-1} \quad ,$$

for some $K > 0$ not depending on s, since d is bounded. Thus (4.7) follows from the second condition in (4.4).

Step 2. We now show Relation (4.5). Let us now define, for all $n \in \mathbb{N}$, the function $\Psi_n : \lambda \mapsto \sum_{k=0}^n (k+1)^{-N} \mathrm{e}^{-\mathrm{i}\lambda k}$, then for all $n \in \mathbb{N}$, for all $t \in \mathbb{Z}$,

$$\left[F_{\Psi_n}(\epsilon) \right]_t = \sum_{k=0}^n (k+1)^{-N} \epsilon_{t-k} \; .$$

Moreover, by Corollary 3.3, the series $\sum_{k=0}^\infty (k+1)^{-N} \epsilon_{t-k}$ converges in $\mathcal{M}(\Omega, \mathcal{G}, \mathcal{H}_0, \mathbb{P})$ for all $t \in \mathbb{Z}$ if and only if the sequence $(\Psi_n)_{n\in\mathbb{N}}$ converges in $\mathsf{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{O}(\mathcal{H}_0), \nu_\epsilon)$ and, in this case, we have for all $t \in \mathbb{Z}$,

$$\sum_{k=0}^\infty (k+1)^{-N} \epsilon_{t-k} = [F_\Psi(\epsilon)]_t \quad \text{in } \mathcal{M}(\Omega, \mathcal{G}, \mathcal{H}_0, \mathbb{P}) \; ,$$

where $\Psi = \lim_{n\to+\infty} \Psi_n$ in $\mathsf{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{O}(\mathcal{H}_0), \nu_\epsilon)$. Suppose such a convergence holds, then (4.5) is equivalent to

$$(1-\mathrm{e}^{-\mathrm{i}\lambda})^{N-\mathrm{Id}} = C\Psi(\lambda) + \sum_{k=0}^\infty \Delta_k \mathrm{e}^{-\mathrm{i}\lambda k} \quad \text{for Leb-almost every } \lambda \in \mathbb{T} \; . \tag{4.8}$$

We apply Lemma 4.4. By the first condition in (4.4), we have $\varrho \ge 1/2$ and thus that $\|\Delta_k\| = O(k^{-3/2})$ (so that the series $\sum_{k=0}^\infty \Delta_k \epsilon_{t-k}$ converges absolutely in $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$) and that (4.8) holds for $\Psi : \lambda \mapsto \sum_{k=0}^\infty (k+1)^{-N} \mathrm{e}^{-\mathrm{i}\lambda k}$ and that Ψ_n converges almost everywhere to Ψ. To conclude the proof, it only remains to show that Ψ_n converges to Ψ in $\mathsf{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{O}(\mathcal{H}_0), \nu_\epsilon)$. Using Parseval's identity in $L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{S}_2(\mathcal{H}_0), \mathrm{Leb})$ and the same ideas as in the previous step, we have for all $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{\mathbb{T}} \left\| \left(\sum_{k=n+1}^{\infty} (k+1)^{-N} \mathrm{e}^{-\mathrm{i}\lambda k} \right) \Gamma_\epsilon(0)^{1/2} \right\|_2^2 \mathrm{d}\lambda &= \sum_{k=n+1}^{\infty} \int_{\mathbb{T}} \left\| (k+1)^{-N} \Gamma_\epsilon(0)^{1/2} \mathrm{e}^{-\mathrm{i}\lambda k} \right\|_2^2 \mathrm{d}\lambda \\
&= \sum_{k=n+1}^{\infty} \int_{\mathbb{T}} \mathbb{E}\left[\left\| M_{(k+1)^{-d}\mathrm{e}^{-\mathrm{i}\lambda k}} U \epsilon_0 \right\|_{L^2(\mathsf{X},\mathcal{X},\mu)}^2 \right] \mathrm{d}\lambda \\
&= 2\pi \sum_{k=n+1}^{\infty} \int_{\mathsf{X}} \sigma_U^2(s) \left| (k+1)^{-d(s)} \right|^2 \mu(\mathrm{d}s) \\
&= 2\pi \int_{\mathsf{X}} \sigma_U^2(s) \sum_{k=n+1}^{\infty} (k+1)^{-2h(s)} \mu(\mathrm{d}s) \\
&\leq 2\pi \int_{\mathsf{X}} \sigma_U^2(s) \frac{(n+1)^{1-2h(s)}}{2h(s)-1} \mu(\mathrm{d}s)
\end{aligned}$$

where the last inequality is obtained from an obvious inequality between series and integral. By the second condition in (4.4), using Lebesgue's dominated convergence theorem, we get that Ψ_n converges to Ψ in $\mathsf{L}^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mathcal{O}(\mathcal{H}_0), \nu_\epsilon)$. ☐

5 Postponed proofs

5.1 Proofs of Section 2

Proof of Lemma 2.1. By (2.3), we only need to show that, if Φ is simply measurable then it is measurable. The space $\mathcal{E}$ is separable because the set of finite rank operators is dense in $\mathcal{E}$ for the norm $\|\cdot\|$ if $\mathcal{E} = \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$ and $\|\cdot\|_p$ if $\mathcal{E} = \mathcal{S}_p(\mathcal{H}_0, \mathcal{G}_0)$. By Pettis's measurability theorem (see [11, Theorem II.1.2]), this implies that it is enough to show that for all $f \in \mathcal{E}^*$, $f \circ \Phi$ is a measurable complex-valued function. By [8, Theorems 19.1, 18.14, 19.2] we get that $\mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)^*$, $\mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)^*$ and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)^*$ are respectively isometrically isomorphic to $\mathcal{S}_1(\mathcal{H}_0, \mathcal{G}_0)$, $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0)$ and the duality relation can be defined on $\mathcal{E} \times \mathcal{E}^*$ as $(\Psi, \Theta) \mapsto \mathrm{Tr}(\Theta^{\mathsf{H}}\Psi)$. This means that we only have to show measurability of the complex-valued functions $t \mapsto \mathrm{Tr}(\Theta^{\mathsf{H}}\Phi(t))$ for all $\Theta \in \mathcal{E}^*$. Let $(\phi_k)_{k\in\mathbb{N}}, (\psi_k)_{k\in\mathbb{N}}$ be Hilbert basis of $\mathcal{H}_0$ and $\mathcal{G}_0$ respectively, then $\mathrm{Tr}(\Theta^{\mathsf{H}}\Phi(t)) = \sum_{k\in\mathbb{N}} \langle \Phi(t)\phi_k, \Theta\psi_k \rangle_{\mathcal{G}_0}$ which defines a measurable function of t by simple measurability of Φ. ☐

Proof of Lemma 2.3. The first point comes from the fact that for all $A \in \mathcal{X}$, $\nu(A) \preceq \nu(\mathsf{X})$. Now, if ν is trace-class, then (2.4) is easily verified for the norm $\|\cdot\|_1$ using the fact that $\|\cdot\|_1 = \mathrm{Tr}(\cdot)$ for positive operators. Finally, by definition of $\|\nu\|_1$, regularity of $\|\nu\|_1$ is equivalent to regularity of ν as a $\mathcal{S}_1(\mathcal{H}_0)$-valued measure which clearly implies regularity of $y^{\mathsf{H}}\nu x$ for all $x, y \in \mathcal{H}_0$. Suppose now that for all $x, y \in \mathcal{H}_0$, $y^{\mathsf{H}}\nu x$ is regular, then let $(e_k)_{k\in\mathbb{N}}$ be a Hilbert-basis of $\mathcal{H}_0$, and define for all $n \in \mathbb{N}$, the non-negative measure $\mu_n := \sum_{k=0}^{n} e_k^{\mathsf{H}}\nu e_k$ such that for all $A \in \mathcal{X}$, $\|\nu\|_1(A) = \lim_{n\to+\infty} \mu_n(A) = \sup_{n\in\mathbb{N}} \mu_n(A)$. Then by Vitali-Hahn-Sakh-Nikodym's theorem (see [7]) the sequence $(\mu_n)_{n\in\mathbb{N}}$ is uniformly countably additive which implies regularity of $\|\nu\|_1$ by [11, Lemma VI.2.13]. ☐

Proof of Theorem 2.4. Suppose $\|\nu\|_1 \ll \mu$, then, since $\mathcal{S}_1(\mathcal{H}_0)$ is separable and is the dual of $\mathcal{K}(\mathcal{H}_0)$, it is a *separable dual space* and [11, theorem III.3.1] gives the existence and uniqueness of a density $g \in L^1(\mathsf{X}, \mathcal{X}, \mathcal{S}_1(\mathcal{H}_0), \mu)$ satisfying (2.5). Then for all $x \in \mathcal{H}_0$, for all $A \in \mathcal{X}$,

$$\int_A \langle g(t)x, x \rangle_{\mathcal{H}_0} \mu(\mathrm{d}t) = \langle \nu(A)x, x \rangle_{\mathcal{H}_0} \geq 0$$

and therefore there exists a set $A_x \in \mathcal{X}$ with $\mu(A) = 0$ and $\langle g(t)x, x \rangle_{\mathcal{H}_0} \geq 0$ for all $t \in A_x^c$. Taking $(x_n)_{n\in\mathbb{N}}$ a dense countable subset of $\mathcal{H}_0$ we get that f is positive on $\left(\bigcup_{n\in\mathbb{N}} A_{x_n}\right)^c$ where $\mu\left(\bigcup_{n\in\mathbb{N}} A_{x_n}\right) = 0$ thus proving Assertion (a). Moreover, taking the trace in (2.5) gives for all $A \in \mathcal{X}$,

$$\|\nu\|_1(A) = \int_A \|g\|_1 \, \mathrm{d}\mu$$

which gives Assertion (b) and implies easily Assertion (c). The converse implication is a consequence of Assertion (b). ☐

Proof of Proposition 2.7. The proof is easily derived from the fact that $g = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}\|g\|_1$ (see Theorem 2.4) and the definition of $\mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Note that $\Phi g \Phi^{\mathsf{H}} \in \mathbb{F}(\mathsf{X}, \mathcal{X}, \mathcal{S}_1(\mathcal{G}_0))$ and $\Phi g^{1/2} \in \mathbb{F}(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0))$ by simple-measurability of Φ and g and Lemma 2.1. $\square$

Proof of Proposition 2.8. All theses results, except Relation (2.12), are easily derived from the characterization of Proposition 2.7 and the module nature of $L^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \mu)$. We now prove Relation (2.12). First note that $\|\nu\|_1(\{g = 0\}) = \int_{\{g=0\}} \|g\|_1 \,\mathrm{d}\mu = 0$ and therefore

$$\begin{aligned}
\|\nu\|_1\left(\{\Phi f^{1/2} \neq 0\}\right) &= \|\nu\|_1\left(\{\Phi f^{1/2} \neq 0\} \cap \{g \neq 0\}\right) \\
&= \|\nu\|_1\left(\left\{\frac{\Phi f^{1/2}}{\|g\|_1^{1/2}} \neq 0\right\} \cap \{g \neq 0\}\right) \\
&= \|\nu\|_1\left(\{\Phi g^{1/2} \neq 0\} \cap \{g \neq 0\}\right) \\
&\leq \|\nu\|_1\left(\{\Phi g^{1/2} \neq 0\}\right)
\end{aligned}$$

which gives inclusion ($\subset$) of (2.12) since $\|\nu\|_1 \ll \mu$. Conversely, if $\|\nu\|_1\left(\{\Phi f^{1/2} \neq 0\}\right) = 0$, then

$$\mu\left(\{\Phi g^{1/2} \neq 0\}\right) = \mu\left(\{\Phi f^{1/2}\|g\|_1^{1/2} \neq 0\}\right) = \mu\left(\{\Phi f^{1/2} \neq 0\} \cap \{g \neq 0\}\right) = 0$$

because $0 = \|\nu\|_1\left(\{\Phi f^{1/2} \neq 0\}\right) = \int_{\{\Phi f^{1/2} \neq 0\}} \|g\|_1 \,\mathrm{d}\mu$. $\square$

Proof of Proposition 2.9. Since $\|\nu\|_1(\{g = 0\}) = 0$ and $g = f\|g\|_1$, where $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$, we get

$$\begin{aligned}
\|\nu\|_1\left(\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\}\right) &= \|\nu\|_1\left(\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g \neq 0\}\right) \\
&= \|\nu\|_1\left(\left\{\mathrm{Im}(g^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g \neq 0\}\right) \\
&\leq \|\nu\|_1\left(\left\{\mathrm{Im}(g^{1/2}) \not\subset \mathcal{D}(\Phi)\right\}\right)
\end{aligned}$$

which gives (i') $\Rightarrow$ (i) since $\|\nu\|_1 \ll \mu$. Conversely, if $\|\nu\|_1\left(\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\}\right) = 0$, then

$$\mu\left(\left\{\mathrm{Im}(g^{1/2}) \not\subset \mathcal{D}(\Phi)\right\}\right) = \mu\left(\left\{\mathrm{Im}(f^{1/2}) \not\subset \mathcal{D}(\Phi)\right\} \cap \{g \neq 0\}\right) = 0$$

because $0 = \|\nu\|_1\left(\left\{\mathrm{Im}(g^{1/2}) \not\subset \mathcal{D}(\Phi)\right\}\right) = \int_{\left\{\mathrm{Im}(g^{1/2}) \not\subset \mathcal{D}(\Phi)\right\}} \|g\|_1 \,\mathrm{d}\mu$. Hence (i') $\Leftrightarrow$ (i).

Moreover, Equivalences (ii) $\Leftrightarrow$ (ii') and (iii) $\Leftrightarrow$ (iii') and Relation (2.14) are easy consequence of the fact that $g = f\|g\|_1$ and the other results come easily using the definition of $\mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Again, note that measurability of $\Phi g^{1/2}$ and $(\Phi g^{1/2})(\Phi g^{1/2})$ are ensured by $\mathcal{O}$-measurability of Φ, simple measurability of f and Lemma 2.1. $\square$

Proof of Theorem 2.10. As for Proposition 2.8, these results come easily using the definition and Identity (2.16). Relation (2.18) is proven the same way as (2.12). $\square$

Proof of Theorem 2.11. In the first two steps of the proof of [20, Theorem 3.4.12], [27, Theorem 4.22] the authors show that, if $\Phi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and $\epsilon > 0$, there exists $\Psi \in L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$ such that $\|\Phi - \Psi\|_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)} < \epsilon$. This implies that $L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ is dense in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. The other results follow using (2.20) and density of simple functions and trigonometric polynomials in $L^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ (see Theorem B.1). $\square$

Proof of Theorem 2.12. For all $A, B \in \mathcal{X}$ and $\Phi, \Psi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$, we have, by Theorem 2.10,

$$\begin{aligned}
[\mathbb{1}_A \Phi, \mathbb{1}_B \Psi]_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_W)} &= \Phi \nu_W(A \cap B) \Psi^{\mathsf{H}} \\
&= \Phi \,\mathrm{Cov}\left(W(A), W(B)\right) \Psi^{\mathsf{H}} \\
&= \mathrm{Cov}\left(\Phi W(A), \Psi W(B)\right) \\
&= [\Phi W(A), \Psi W(B)]_{\mathcal{G}} \,.
\end{aligned}$$

Then Proposition 2.5, applied to $J = \mathcal{X}\times\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0)$ with $v_{(A,\Phi)} = \mathbb{1}_A\Phi$ and $w_{(A,\Phi)} = \Phi W(A)$, gives that there exists a unique gramian-isometry

$$I_W^{\mathcal{G}_0} : \overline{\mathrm{Span}}^{\mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu_W)} \left(\mathbb{1}_A\Psi\Phi \;:\; A\in\mathcal{X},\Phi\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\Psi\in\mathcal{L}_b(\mathcal{G}_0)\right) \to \mathcal{G}$$

such that for all $A\in\mathcal{X},\Phi\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0)$, $I_W^{\mathcal{G}}(\mathbb{1}_A\Phi) = \Phi W(A)$ and, in addition,

$$\mathrm{Im}(I_W^{\mathcal{G}_0}) = \overline{\mathrm{Span}}^{\mathcal{G}} \left(\Psi\Phi W(A) \;:\; A\in\mathcal{X},\Phi\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\Psi\in\mathcal{L}_b(\mathcal{G}_0)\right) . \tag{5.1}$$

Now, note that

$$\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0) = \{\Psi\Phi \;:\; \Phi\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\Psi\in\mathcal{L}_b(\mathcal{G}_0)\} . \tag{5.2}$$

This gives

$$\mathrm{Span}\left(\mathbb{1}_A\Psi\Phi \;:\; A\in\mathcal{X},\Phi\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\Psi\in\mathcal{L}_b(\mathcal{G}_0)\right) = \mathrm{Span}\left(\mathbb{1}_A\Phi \;:\; A\in\mathcal{X},\Phi\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0)\right)$$

and therefore, by Theorem 2.11,

$$\overline{\mathrm{Span}}^{\mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu_W)} \left(\mathbb{1}_A\Psi\Phi \;:\; A\in\mathcal{X},\Phi\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0),\Psi\in\mathcal{L}_b(\mathcal{G}_0)\right) = \mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{O}(\mathcal{H}_0,\mathcal{G}_0),\nu_W) .$$

Finally, (5.2) with (5.1) gives

$$\mathrm{Im}(I_W^{\mathcal{G}_0}) = \overline{\mathrm{Span}}^{\mathcal{G}} \left(\Phi W(A) \;:\; A\in\mathcal{X},\Phi\in\mathcal{L}_b(\mathcal{H}_0,\mathcal{G}_0)\right)$$

which concludes the proof. □

Proof of Theorem 2.13. It is easily seen that the mapping

$$W : \begin{array}{ccc} \mathcal{X} & \to & \mathcal{H} \\ A & \mapsto & w(\mathbb{1}_A \mathrm{Id}_{\mathcal{H}_0}) \end{array}$$

defines a c.a.g.o.s. measure on $(\mathsf{X},\mathcal{X},\mathcal{H})$ with intensity operator measure ν. Then

$$\begin{aligned}\mathrm{Cov}\left(W(A),W(B)\right) = [w(\mathbb{1}_A\mathrm{Id}_{\mathcal{H}_0}),w(\mathbb{1}_B\mathrm{Id}_{\mathcal{H}_0})]_{\mathcal{H}} &= [\mathbb{1}_A\mathrm{Id}_{\mathcal{H}_0},\mathbb{1}_B\mathrm{Id}_{\mathcal{H}_0}]_{\mathsf{L}^2(\mathsf{X},\mathcal{X},\mathcal{O}(\mathcal{H}_0),\nu)} \\ &= \nu(A\cap B).\end{aligned}$$

Moreover, we have for all $A\in\mathcal{X}$, $\Phi\in\mathcal{L}_b(\mathcal{H}_0)$, $\Phi W(A) = \Phi w(\mathbb{1}_A\mathrm{Id}) = w(\mathbb{1}_A\Phi\mathrm{Id}) = w(\mathbb{1}_A\Phi)$. Hence by Theorem 2.12, (2.22) holds, which uniquely defines W. □

5.2 Proofs of Section 3

Proof of Theorem 3.2. If X is weakly-stationary then, by Lemma 3.1, the family of shifts $(U_h^X)_{h\in\mathrm{T}}$ is a c.g.u.r. of T on $\mathcal{H}^X$. Hence Theorem 2.6 gives that there exists a unique regular gramian-projection valued measure ξ^X on $(\hat{\mathrm{T}},\mathcal{B}(\hat{\mathrm{T}}),\mathcal{H}^X)$ such that, for all $h\in\mathrm{T}$,

$$U_h^X = \int_{\hat{\mathrm{T}}} \chi(h)\,\xi^X(\mathrm{d}\chi) . \tag{5.3}$$

The mapping

$$\hat{X} : \begin{array}{ccc} \mathcal{B}(\hat{\mathrm{T}}) & \to & \mathcal{H} \\ A & \mapsto & \xi^X(A)X_0 \end{array}$$

is then a c.a.g.o.s. measure on $(\hat{\mathrm{T}},\mathcal{B}(\hat{\mathrm{T}}),\mathcal{H}^X)$ and is regular because for all $Y\in\mathcal{H}^X$, $\left\langle \hat{X}(\cdot),Y\right\rangle_{\mathcal{H}^X} = Y^{\mathsf{H}}\xi^X(\cdot)X_0$ is regular. Since $\mathcal{H}^X$ is gramian isometrically embedded in $\mathcal{H}$, $\hat{X}$ is also a regular c.a.g.o.s. measure on $(\hat{\mathrm{T}},\mathcal{B}(\hat{\mathrm{T}}),\mathcal{H})$. Then, from (5.3) we get that for all $t\in\mathrm{T}$,

$$X_t = U_t^X X_0 = \int_{\hat{\mathrm{T}}} \chi(t)\,\xi^X(\mathrm{d}\chi)X_0 = \int_{\hat{\mathrm{T}}} \chi(t)\,\hat{X}(\mathrm{d}\chi) .$$

To show uniqueness, suppose there exists another regular c.a.g.o.s. measure W on $(\hat{\mathrm{T}},\mathcal{B}(\hat{\mathrm{T}}),\mathcal{H})$ satisfying (3.1), then the gramian-isometries $I_{\hat{X}}^{\mathcal{H}_0}$ and $I_W^{\mathcal{H}_0}$ coincide on $\mathrm{Span}\left(\chi\mapsto\chi(t)\Phi \,:\, t\in\mathrm{T},\Phi\in\mathcal{L}_b(\mathcal{H}_0)\right)$ and therefore, on $\mathsf{L}^2(\hat{\mathrm{T}},\mathcal{B}(\hat{\mathrm{T}}),\mathcal{O}(\mathcal{H}_0),\nu_X)\,\cap$

$\mathsf{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{H}_0), \nu_W)$ by Theorem 2.11. In particular for all $A \in \mathcal{X}$ we get $W(A) = I_W^{\mathcal{H}_0}(\mathbb{1}_A \mathrm{Id}_{\mathcal{H}_0}) = I_{\hat{X}}^{\mathcal{H}_0}(\mathbb{1}_A \mathrm{Id}_{\mathcal{H}_0}) = \hat{X}(A)$.

Relation (3.2) follows from (3.1) and the gramian-isometric property of $I_{\hat{X}}^{\mathcal{H}_0}$ and uniqueness of ν_X comes from (3.2) and Theorem B.1.

Finally, we show the converse statement in Theorem 3.2. Suppose that there exists a regular c.a.g.o.s. $\hat{X}$ on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{H})$ satisfying (3.1). Then, the first two points of Definition 1.3 are straightforward and, calling ν_X the intensity operator measure of $\hat{X}$ and using the gramian-isometric property of integration with respect to $\hat{X}$, we get for all $t, h \in \mathrm{T}$, $\mathrm{Cov}\,(X_{t+h}, X_t) = \int_{\hat{\mathrm{T}}} \chi(t+h)\overline{\chi(t)}\nu_X(\mathrm{d}\chi) = \int_{\hat{\mathrm{T}}} \chi(h)\nu_X(\mathrm{d}\chi)$ which gives the third point of Definition 1.3. Finally, for all $\Phi \in \mathcal{L}_b(\mathcal{H}_0)$, for all $h \in \mathrm{T}$,

$$\mathrm{Tr}(\Phi\Gamma(h)) = \mathrm{Tr}\left(\Phi \int_{\hat{\mathrm{T}}} \chi(h)\, \nu_X(\mathrm{d}\chi)\right) = \int_{\hat{\mathrm{T}}} \chi(h) \mathrm{Tr}(\Phi f_{\nu_X}(\chi))\, \|\nu_X\|_1(\mathrm{d}\chi)$$

and, since for $\|\nu_X\|_1$-almost every $\chi \in \hat{\mathrm{T}}$, $|\mathrm{Tr}(\Phi f_{\nu_X}(\chi))| \le \|\Phi\| \|f_{\nu_X}(\chi)\|_1 = \|\Phi\|$, we get continuity of $h \mapsto \mathrm{Tr}(\Phi\Gamma(h))$ by Lebesgue's dominated convergence theorem thus showing the last point of Definition 1.3. □

Proof of Corollary 3.3. By gramian-isometry of integration with respect to $\hat{X}$, we get for all $h, t \in \mathrm{T}$, for all $\Phi, \Psi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$,

$$[\Phi X_h, \Psi X_t]_{\mathcal{H}} = \left[\int_{\hat{\mathrm{T}}} \chi(h)\Phi\, \hat{X}(\mathrm{d}\chi), \int_{\hat{\mathrm{T}}} \chi(t)\Psi\, \hat{X}(\mathrm{d}\chi)\right]_{\mathcal{H}} = [\mathrm{e}_h\Phi, \mathrm{e}_t\Psi]_{\mathsf{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X)}\,. \tag{5.4}$$

where $\mathrm{e}_t : \chi \mapsto \chi(t)$ for all $t \in \mathrm{T}$. Then, by Proposition 2.5 and Theorem 2.11, there is a unique gramian-isometry

$$I : \mathsf{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu_X) \to \mathcal{G}$$

which maps $\mathrm{e}_t\Phi$ to ΦX_t for all $t \in \mathrm{T}, \Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $\mathrm{Im}(I) = \mathcal{H}^{X, \mathcal{G}_0}$. Since the mapping $I_{\hat{X}}^{\mathcal{G}_0} : \Phi \mapsto \int_{\hat{\mathrm{T}}} \Phi\, \mathrm{d}\hat{X}$ satisfies this condition we have $I_{\hat{X}}^{\mathcal{G}_0} = I$ thus concluding the proof. □

Before proving the results on composition and inversion of filters, let us prove a useful lemma.

Lemma 5.1. *Let $\mathcal{H}_0$, $\mathcal{G}_0$, $\mathcal{I}_0$ be separable Hilbert spaces and $A \in \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0)$, $B \in \mathcal{K}(\mathcal{H}_0, \mathcal{G}_0)$. The following assertions hold.*

(i) $\mathrm{Im}(|B^{\mathsf{H}}|) = \mathrm{Im}(B)$.

(ii) If $\mathrm{Im}(B) \subset \mathcal{D}(A)$, *then* $(AB)(AB)^{\mathsf{H}} = (A|B^{\mathsf{H}}|)(A|B^{\mathsf{H}}|)^{\mathsf{H}}$.

(iii) If $\mathrm{Im}(B) \subset \mathcal{D}(A)$, *then* $AB \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$ *if and only if* $A|B^{\mathsf{H}}| \in \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0)$. *In this case* $\|AB\|_2 = \|A|B^{\mathsf{H}}|\|_2$.

Proof. Let us consider the singular values decomposition $B = \sum_{n \in \mathbb{N}} \sigma_n \psi_n \otimes \phi_n$, then $|B^{\mathsf{H}}| = \sum_{n \in \mathbb{N}} \sigma_n \psi_n \otimes \psi_n$.

Proof of (i). We have $\mathrm{Im}(B) = \left\{\sum_{n \in \mathbb{N}} \sigma_n x_n \psi_n \,:\, (x_n)_{n \in \mathbb{N}} \in \ell^2(\mathbb{N})\right\} = \mathrm{Im}(|B^{\mathsf{H}}|)$.

Proof of (ii). By the first point both composition AB and $A|B^{\mathsf{H}}|$ make sense. Consider the polar decomposition of B^{H} : $B^{\mathsf{H}} = U|B^{\mathsf{H}}|$, then $B = |B^{\mathsf{H}}|U^{\mathsf{H}}$ and we get

$$(AB)(AB)^{\mathsf{H}} = (A\left|B^{\mathsf{H}}\right|)U^{\mathsf{H}}U(A\left|B^{\mathsf{H}}\right|)^{\mathsf{H}} = (A\left|B^{\mathsf{H}}\right|)P_{\ker(|B^{\mathsf{H}}|)^{\perp}}(A\left|B^{\mathsf{H}}\right|)^{\mathsf{H}} = (A\left|B^{\mathsf{H}}\right|)(A\left|B^{\mathsf{H}}\right|)^{\mathsf{H}}$$

because $|B^{\mathsf{H}}|P_{\ker(|B^{\mathsf{H}}|)^{\perp}} = |B^{\mathsf{H}}|$.

Proof of (iii). We have that $AB \in \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0)$ if and only if $(AB)(AB)^{\mathsf{H}} \in \mathcal{S}_1(\mathcal{I}_0)$, which is equivalent to $A|B^{\mathsf{H}}| \in \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0)$ by the previous point. □

Proof of Theorem 3.4. Let μ be a dominating measure for $\|\nu\|_1$ and $g = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$, then, by definition of $\Phi\nu\Phi^{\mathsf{H}}$, μ also dominates $\|\Phi\nu\Phi^{\mathsf{H}}\|_1$ and $\frac{\mathrm{d}\Phi\nu\Phi^{\mathsf{H}}}{\mathrm{d}\mu} = (\Phi g^{1/2})(\Phi g^{1/2})^{\mathsf{H}}$. Hence,

$\left(\frac{\mathrm{d}\Phi\nu\Phi^{\mathsf{H}}}{\mathrm{d}\mu}\right)^{1/2} = \left|(\Phi g^{1/2})^{\mathsf{H}}\right|$ and we get, by Proposition 2.9,

$$\Psi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}}) \Leftrightarrow \begin{cases} \mathrm{Im}\left|(\Phi g^{1/2})^{\mathsf{H}}\right| \subset \mathcal{D}(\Psi) \ \ \mu\text{-a.e.} \\ \Psi\left|(\Phi g^{1/2})^{\mathsf{H}}\right| \in \mathcal{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{G}_0, \mathcal{I}_0), \mu) \end{cases}$$
$$\Leftrightarrow \begin{cases} \mathrm{Im} g^{1/2} \subset \mathcal{D}(\Psi\Phi) \ \ \mu\text{-a.e.} \\ \Psi\Phi g^{1/2} \in \mathcal{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{I}_0), \mu) \end{cases}$$
$$\Leftrightarrow \Psi\Phi \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu)$$

where the second equivalence comes from Lemma 5.1 and the fact that $t \in \mathsf{X}$, $\mathcal{D}(\Psi(t)\Phi(t)) = \Phi(t)^{-1}(\mathcal{D}(\Psi(t)))$ which gives that $\mathrm{Im}(g^{1/2}(t)) \subset \mathcal{D}(\Psi(t)\Phi(t))$ if and only if $\mathrm{Im}(\Phi(t)g^{1/2}(t)) \subset \mathcal{D}(\Psi(t))$. Moreover, Assertion (a) holds because for all $\Psi, \Theta \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \Phi\nu\Phi^{\mathsf{H}})$ and $A \in \mathcal{X}$,

$$\begin{aligned} (\Psi\Phi)\nu(\Theta\Phi)^{\mathsf{H}}(A) &= \int_A \left(\Psi\Phi g^{1/2}\right)\left(\Theta\Phi g^{1/2}\right)^{\mathsf{H}} \mathrm{d}\mu \\ &= \int_A \left(\Psi\left|(\Phi g^{1/2})^{\mathsf{H}}\right|\right)\left(\Theta\left|(\Phi g^{1/2})^{\mathsf{H}}\right|\right)^{\mathsf{H}} \mathrm{d}\mu \qquad \text{(by lemma 5.1)} \\ &= \Psi(\Phi\nu\Phi^{\mathsf{H}})\Theta^{\mathsf{H}}(A) \end{aligned}$$

which also gives Assertion (b) by taking $A = \mathsf{X}$. Finally, to show Assertion (c), suppose that Φ is injective $\|\nu\|_1$-a.e. then $\Phi^{-1}\Phi : \lambda \mapsto \mathrm{Id}_{\mathcal{H}_0}\mathbb{1}_{\{\Phi(\lambda) \text{ is injective}\}}$ is in $\mathscr{L}^2(\mathrm{T}, \mathcal{B}(\mathrm{T}), \mathcal{O}(\mathcal{H}_0), \nu)$ which gives that $\Phi^{-1} \in \mathscr{L}^2(\mathrm{T}, \mathcal{B}(\mathrm{T}), \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \Phi\nu\Phi^{\mathsf{H}})$ by Assertion (a). □

Proof of Corollary 3.5. This is a clear consequence of Assertion (b) of Theorem 3.4 and Theorem 2.12. □

Proof of Corollary 3.6. If $W \in \hat{\mathcal{S}}_\Phi$, then the equivalence between $W \in \hat{\mathcal{S}}_{\Psi\Phi}$ and $\hat{F}_\Phi(W) \in \hat{\mathcal{S}}_\Psi$ is just another formulation of the equivalence (3.4). Moreover, as explained in Remark 3.1, showing (3.6) is equivalent to showing (3.7) with $V = \hat{F}_\Phi(W)$ and $A \in \mathcal{X}$.

We first prove (3.7) for Ψ of the form $\Psi = \Theta\mathbb{1}_B$ with $\Theta \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$ and $B \in \mathcal{X}$. Calling $C = A \cap B$, (3.7) becomes

$$\Theta\left(\int_C \Phi(\lambda)\, W(\mathrm{d}\lambda)\right) = \int_C \Theta\Phi(\lambda)\, W(\mathrm{d}\lambda)\,. \tag{5.5}$$

When $\mathcal{I}_0 = \mathcal{G}_0$ this identity comes from the fact that the integral with respect to W is $\mathcal{L}_b(\mathcal{G}_0)$-linear. When $\mathcal{I}_0 \neq \mathcal{G}_0$, we have to show it by hand. Using the notations $I_W^{\mathcal{G}_0}$ and $I_W^{\mathcal{I}_0}$, Relation (5.5) is equivalent to

$$I_W^{\mathcal{I}_0}(\mathbb{1}_C\Theta\Phi) = \Theta I_W^{\mathcal{G}_0}(\mathbb{1}_C\Phi) \tag{5.6}$$

If Φ is of the type $\Phi = \Lambda\mathbb{1}_D$ with $\Lambda \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ and $D \in \mathcal{X}$, then $\Theta\Lambda \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{I}_0)$ and we immediately get

$$I_W^{\mathcal{I}_0}(\Theta\Phi) = I_W^{\mathcal{I}_0}(\Theta\Lambda\mathbb{1}_D) = \Theta\Lambda W(D) = \Theta I_W^{\mathcal{G}_0}(\Lambda\mathbb{1}_D) = \Theta I_W^{\mathcal{G}_0}(\Phi)$$

This property extends to the case Φ is a simple $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued function by linearity and if $\Phi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$, Theorem 2.11 gives that there exists a sequence $(\Phi_n)_{n\in\mathbb{N}}$ of simple $\mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$-valued functions converging to Φ in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Hence, calling $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$, we get

$$\begin{aligned} \|\Theta\Phi - \Theta\Phi_n\|^2_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu)} &= \int \left\|\Theta(\Phi - \Phi_n)f^{1/2}\right\|_2^2 \mathrm{d}\|\nu\|_1 \\ &\leq \|\Theta\| \int \left\|(\Phi - \Phi_n)f^{1/2}\right\|_2^2 \mathrm{d}\|\nu\|_1 \\ &= \|\Theta\| \|\Phi - \Phi_n\|^2_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{G}_0), \nu)} \\ &\xrightarrow[n\to+\infty]{} 0\,. \end{aligned}$$

Since for all $n \in \mathbb{N}$, $I_W^{\mathcal{I}_0}(\Theta\Phi_n) = I_W^{\mathcal{G}_0}(\Phi_n)$ and by continuity of $I_W^{\mathcal{I}_0}$ and $I_W^{\mathcal{G}_0}$, we finally get (5.6), that is (3.7) for $V = \hat{F}_\Psi(W)$ where Ψ has the form $\Psi = \Theta\mathbb{1}_B$ with $\Theta \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$

and $B \in \mathcal{X}$. By linearity, it follows that (3.7) still holds with $V = \hat{F}_\Phi(W)$ and all simple $\mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$-valued function Ψ.

Finally, if $\Psi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu)$, then, by Theorem 2.11, there exists a sequence $(\Psi_n)_{n\in\mathbb{N}}$ of simple $\mathcal{L}_b(\mathcal{G}_0, \mathcal{I}_0)$-valued functions converging to Ψ in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu)$. Since $\Psi \mapsto \Psi\Phi$ is a gramian-isometry from $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{I}_0), \nu_V)$ to $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu_W)$ (see Theorem 3.4), the sequence $(\Psi_n\Phi)_{n\in\mathbb{N}}$ then converges to $\Psi\Phi$ in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{H}_0, \mathcal{I}_0), \nu_W)$ and by continuity of the stochastic integral we get

$$\int_A \Psi \mathrm{d}V = \lim_{n\to+\infty} \int_A \Psi_n \mathrm{d}V = \lim_{n\to+\infty} \int_A \Psi_n \Phi \mathrm{d}W = \int_A \Psi\Phi \mathrm{d}W .$$

□

Proof of Corollary 3.7. As usual, we call $\nu_V = \Phi\nu\Phi^{\mathsf{H}}$ the spectral operator measure of V. Supposing that Φ is injective $\|\nu_W\|_1$-a.e, Assertions (c) and (a) of Theorem 3.4, give that $\Phi^{-1} \in \mathscr{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0), \nu_V)$ (i.e. $V \in \hat{\mathcal{S}}_{\Phi^{-1}}$) and $\Phi^{-1}\nu_V \left(\Phi^{-1}\right)^{\mathsf{H}} = \nu_W$. Hence, writing Relation (3.6) with $\Psi = \Phi^{-1}$, we get $\hat{F}_{\Phi^{-1}}(V) = \hat{F}_{\Phi^{-1}\Phi}(W) = W$. Then, reversing the roles of W and V in Corollary 3.5 gives the reciprocal $\underset{\approx}{\supseteq}$ in (3.5). □

6 Additional comments

6.1 Bochner's and Stone's theorems and their generalizations to normal Hilbert modules

In the following, we consider an l.c.a. group $(\mathrm{T}, +)$ and a Hilbert space $\mathcal{H}_0$. We discuss here the relations between Bochner's and Stone's theorem and their generalizations for the functional case.

Definition 6.1 (Hermitian non-negative definite function)**.** *A function $\gamma : \mathrm{T} \to \mathbb{C}$ defined on an l.c.a. group $(\mathrm{T}, +)$ is said to be hermitian non-negative definite if for all $n \in \mathbb{N}$, $t_1, \cdots, t_n \in \mathrm{T}$ and $a_1, \cdots, a_n \in \mathbb{C}$,*

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \gamma(t_i - t_j) \geq 0.$$

Theorem 6.1 (Bochner)**.** *Let T be an l.c.a. group and $\gamma : \mathrm{T} \to \mathbb{C}$ be a continuous hermitian non-negative definite function. Then there exists a unique regular finite non-negative measure μ on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$ such that*

$$\gamma(h) = \int_{\hat{\mathrm{T}}} \chi(h)\, \mu(\mathrm{d}\chi), \quad h \in \mathrm{T}. \tag{6.1}$$

Theorem 6.2 (Stone)**.** *Let T be an l.c.a. group and $U : \begin{array}{ccc} \mathrm{T} & \mapsto & \mathcal{L}_b(\mathcal{H}_0) \\ t & \mapsto & U_t \end{array}$ be a c.u.r. of T on a Hilbert space $\mathcal{H}_0$. Then there exists a unique regular projection-valued measure ξ on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}))$ such that*

$$U_h = \int_{\hat{\mathrm{T}}} \chi(h)\, \xi(\mathrm{d}\chi), \quad h \in \mathrm{T}. \tag{6.2}$$

Stone's theorem can be seen as a generalization of Bochner's theorem for operator-valued functions. However, it is not necessary to restrict ourselves to unitary representations of T on $\mathcal{H}_0$ and, using an appropriate definition for hermitian non-negative definite operator-valued functions, one can show that Bochner's theorem still holds. We introduce the two following definitions which will be proved to be equivalent.

Definition 6.2 (Hermitian non-negative definite operator-valued function)**.** *Let $(\mathrm{T}, +)$ be an l.c.a. group and $\mathcal{H}_0$ a Hilbert space. Then a function $\Gamma : \mathrm{T} \to \mathcal{L}_b(\mathcal{H}_0)$ is said to be hermitian non-negative definite if for all $n \in \mathbb{N}$, $t_1, \cdots, t_n \in \mathrm{T}$ and $a_1, \cdots, a_n \in \mathbb{C}$,*

$$\sum_{i,j=1}^{n} a_i \overline{a_j} \Gamma(t_i - t_j) \succeq 0.$$

Equivalently, Γ is hermitian non-negative definite if and only if for all $x \in \mathcal{H}_0$, $t \mapsto \langle \Gamma(t)x, x\rangle_{\mathcal{H}_0}$ is hermitian non-negative definite.

Definition 6.3 (Positive-type operator-valued function)**.** *Let* $(\mathrm{T},+)$ *be an l.c.a. group and* $\mathcal{H}_0$ *a Hilbert space. Then a function* $\Gamma : \mathrm{T} \to \mathcal{L}_b(\mathcal{H}_0)$ *is said to be of positive-type if for all* $n \in \mathbb{N}$*,* $t_1, \cdots, t_n \in \mathrm{T}$ *and* $x_1, \cdots, x_n \in \mathcal{H}_0$*,*

$$\sum_{i,j=1}^{n} \langle \Gamma(t_i - t_j)x_i, x_j\rangle_{\mathcal{H}_0} \geq 0.$$

It is straightforward to see that a positive-type operator-valued function is hermitian non-negative definite. The other implication is not as easy to prove and will be discussed below. Note that unitary representations are hermitian non-negative definite and therefore Stone's theorem is, indeed, a generalization of Bochner's theorem for a particular type of hermitian non-negative definite operator-valued functions. As a full generalization, the following theorem holds.

Theorem 6.3. *Let* $(\mathrm{T},+)$ *be an l.c.a. group,* $\mathcal{H}_0$ *a Hilbert space and* $\Gamma : \mathrm{T} \to \mathcal{L}_b(\mathcal{H}_0)$ *continuous for the w.o.t. Then the following propositions are equivalent*

(i) Γ *is hermitian non-negative definite.*

(ii) Γ *is of positive-type.*

(iii) There exists a regular p.o.v.m. ν *on* $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{H}_0)$ *such that*

$$\Gamma(h) = \int_{\hat{\mathrm{T}}} \chi(h)\, \nu(\mathrm{d}\chi), \quad h \in \mathrm{T}. \tag{6.3}$$

Moreover ν *is the unique regular p.o.v.m. satisfying* (6.3)*.*

These results, as well as Stone's theorem for normal Hilbert modules (see Theorem 2.6) can be proven in different ways, each of which exhibits a specific interest. They also emphasize close relations between these concepts as it turns out that almost every result can be obtained as a consequence of any of the others. As a summary, Figure 1 gives a graphical representation of some interesting implications found in the literature. Arrows with the same color indicate a path of implications usually followed by one or several authors. A few comments on such paths are needed.

- Bochner's and Stone's theorems can be derived on their own using Fourier theory and Riesz-Markov's representation theorem.
- The proofs of Bochner's theorem from Stone's theorem (in cyan) and Naimark's moment theorem from the generalization of Stone's theorem (in brown) use very similar concepts.
- These concepts are closely related to dilation theory (see [37, 3], [16, Section 8]) which is used in [29] to prove Naimark's moment theorem (in green).
- A particular proof of Stone's theorem from Bochner's theorem (in blue) is common in the literature. The proof consists in showing (1.6) when Γ is an u.r. and then proving that the p.o.v.m. ν obtained is actually a projection-valued measure. In fact, the hypothesis that Γ is an u.r. is only useful to show that ν is projection-valued and not to show (1.6). This means that this proof contains a proof of Bochner's theorem for operators as we explicitly represented in blue.
- Concerning the generalization of Bochner's theorem (Theorem 6.3), two results can be found depending on the hypothesis made on the function Γ (hermitian non-negative definite or of positive type). The most general is (i) $\Rightarrow$ (iii) and it is proven (as discussed in the previous point) in a simpler way (without using modules nor dilation theory) than the other implication ((ii) $\Rightarrow$ (iii)). The converse implications are often omitted or stated without proof and the equivalence of Theorem 6.3 is not common in the literature, but can be found in [3]. The implication (iii) $\Rightarrow$ (i) is easily verified using simple properties of p.o.v.m. but (iii) $\Rightarrow$ (ii) does not seem trivial to show. In [3, Theorem 2], the author provides a proof which makes use of dilation theory. This can be avoided using the fact that, if ν is a p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$, then for all $n \in \mathbb{N}^*$, and $x_1, \cdots, x_n \in \mathcal{H}_0$, the mapping

$$\mu : A \mapsto \begin{bmatrix} \langle \nu(A)x_1, x_1\rangle_{\mathcal{H}_0} & \cdots & \langle \nu(A)x_n, x_1\rangle_{\mathcal{H}_0} \\ \vdots & \ddots & \vdots \\ \langle \nu(A)x_1, x_n\rangle_{\mathcal{H}_0} & \cdots & \langle \nu(A)x_n, x_n\rangle_{\mathcal{H}_0} \end{bmatrix}$$

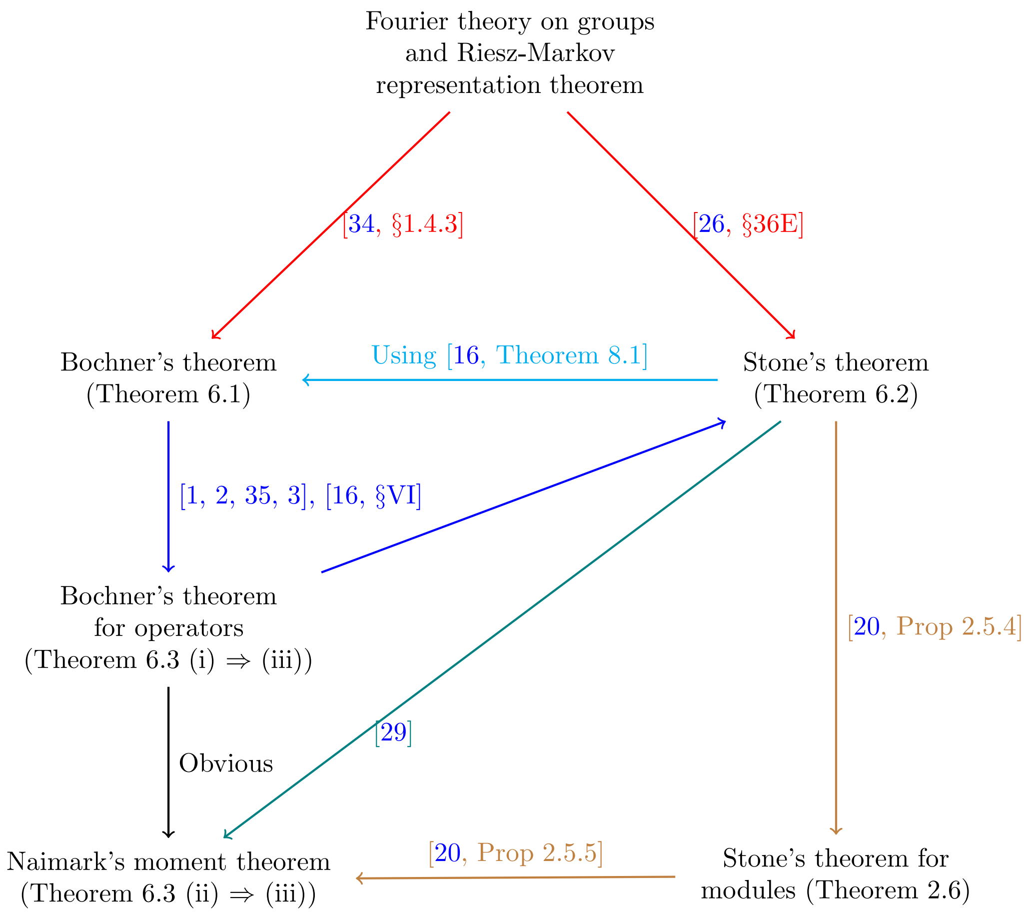

Figure 1: Possible proof paths between the principal results and related concepts.

defines a p.o.v.m. on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathbb{C}^{n\times n})$ (i.e. a hermitian non-negative matrix valued measure). Then, using the results of [33, Section 2] we get that for all $i, j \in [\![1, n]\!]$, $\mu_{i,j} : A \mapsto [\mu(A)]_{i,j}$ admits a density $f_{i,j}$ with respect to the non-negative finite measure $\|\mu\|_1 : A \mapsto \|\mu(A)\|_1 = \mathrm{Tr}(\mu(A))$ and that the matrix-valued function $f : \chi \mapsto (f_{i,j}(\chi))_{1\le i,j\le n}$ is $\|\mu\|_1$-a.e. hermitian, non-negative. Using this, if $\Gamma : h \mapsto \int_{\hat{\mathrm{T}}} \chi(h)\,\nu(\mathrm{d}\chi)$, we get for all $n \in \mathbb{N}^*$, $t_1, \cdots, t_n \in \mathrm{T}$ and $x_1, \cdots, x_n \in \mathcal{H}_0$

$$
\begin{aligned}
\sum_{i,j=1}^{n} \langle \Gamma(t_i - t_j)x_i, x_j\rangle_{\mathcal{H}_0} &= \sum_{i,j=1}^{n} \int_{\hat{\mathrm{T}}} \chi(t_i)\overline{\chi(t_j)}\,\mu_{i,j}(\mathrm{d}\chi) \\
&= \sum_{i,j=1}^{n} \int_{\hat{\mathrm{T}}} \chi(t_i)\overline{\chi(t_j)} f_{i,j}(\chi)\,\|\mu\|_1(\mathrm{d}\chi) \\
&= \int_{\hat{\mathrm{T}}} \underbrace{\sum_{i,j=1}^{n} \chi(t_i)\overline{\chi(t_j)} f_{i,j}(\chi)}_{\ge 0\ \|\mu\|_1\text{-a.e.}}\,\|\mu\|_1(\mathrm{d}\chi) \\
&\ge 0\,.
\end{aligned}
$$

6.2 An alternative construction of spectral representation for functional weakly-stationary processes

In Section 6.1, we saw that Bochner's theorem can be generalized to operator-valued non-negative definite functions. This result can be used to get the same results as in Theorem 3.2 but in a different order. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathrm{T}, +)$ an l.c.a. group. Let $\mathcal{H}_0$ be a separable Hilbert space and set $\mathcal{H} = \mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$. Let $X := (X_t)_{t\in\mathrm{T}}$ be a centered weakly stationary $\mathcal{H}_0$-valued process. Then it is easy to verify that the autocovariance operator function Γ_X is hermitian non-negative definite and continuous for the w.o.t. Hence, by Theorem 6.3, there exists a unique regular p.o.v.m. ν_X of $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{H}_0)$ which satisfies (3.2). Since $\nu_X(\hat{\mathrm{T}}) = \Gamma(0) \in \mathcal{S}_1(\mathcal{H}_0)$, ν_X is a trace-class p.o.v.m. by Lemma 2.3. Now, call $\mathrm{e}_t : \chi \mapsto \chi(t)$ for all $t \in \mathrm{T}$, then, for all $h, t \in \mathrm{T}$, for all $\Phi, \Psi \in \mathcal{L}_b(\mathcal{H}_0)$,

$$[\Phi X_h, \Psi X_t]_{\mathcal{H}} = \Phi\Gamma(h-t)\Psi^{\mathsf{H}} = \Phi\left(\int_{\hat{\mathrm{T}}} \chi(h-t)\nu(\mathrm{d}\chi)\right)\Psi^{\mathsf{H}} = [\mathrm{e}_h\Phi, \mathrm{e}_t\Psi]_{\mathsf{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{H}_0), \nu_X)} \; .$$

Then, by Proposition 2.5 and Theorem 2.11, there is a unique gramian-isometry

$$I : \mathsf{L}^2(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{O}(\mathcal{H}_0), \nu_X) \to \mathcal{H}$$

which maps $\mathrm{e}_t\Phi$ to ΦX_t for all $t \in \mathrm{T}, \Phi \in \mathcal{L}_b(\mathcal{H}_0)$ and $\mathrm{Im}(I) = \mathcal{H}^X$. Using Theorem 2.13, we get that there exists a unique c.a.g.o.s. measure $\hat{X}$ on $(\hat{\mathrm{T}}, \mathcal{B}(\hat{\mathrm{T}}), \mathcal{H})$ with intensity operator measure ν_X such that for $I_{\hat{X}}^{\mathcal{H}_0} = I$. In particular, Relation (3.1) holds and $\hat{X}$ is regular because ν_X is a regular trace class p.o.v.m.

Remark 6.1. *In link with Remark 1.3, it is interesting to note that, in this proof, we use a milder notion of continuity for Γ_X (continuity for the s.o.t.). In fact, the last part of the proof of Theorem 3.2 shows that, in order to have weak-continuity of Γ_X, it is enough to have Relation* (3.2) *which can be obtained using only continuity for the s.o.t. We can therefore state the two following results*

1. *A hermitian non-negative definite operator-valued function $\Gamma : \mathrm{T} \to \mathcal{L}_b(\mathcal{H}_0)$ such that $\Gamma(0) \in \mathcal{S}_1(\mathcal{H}_0)$ is weakly continuous if and only if it is continuous for the s.o.t.*
2. *An L^2, $\mathcal{H}_0$-valued process $X = (X_t)_{t\in\mathrm{T}}$ is weakly-stationary if and only if for all $x \in \mathcal{H}_0$, the L^2, complex-valued process $(\langle X_t, x\rangle_{\mathcal{H}_0})_{t\in\mathrm{T}}$ is weakly-stationary.*

6.3 Comparison with recent approaches

Recently, **R1**, **R2** and the problem of defining filtering in the spectral domain have been addressed for the case $\mathrm{T} = \mathbb{Z}$ in [38] under additional assumptions. An attempt at relaxing these assumption was proposed in [40]. We list here and comment the principal results on spectral analysis presented in [38], [40].

About **R1** : With the additional assumption that $\sum_{h\in\mathbb{Z}} \|\Gamma_X(h)\| < +\infty$, [38, Proposition 2.3.5] states that **R1** holds with $\nu_X(\mathrm{d}\lambda) = f_X(\lambda)\mathrm{d}\lambda$ where

$$f_X(\lambda) = \frac{1}{2\pi}\sum_{t\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}t\lambda}\Gamma_X(t) \in \mathcal{S}_1^+(\mathcal{H}_0) \; ,$$

where the series converges in $\|\cdot\|$. This result restricts the whole spectral theory to the case where the spectral operator measure admits a density with respect to Lebesgue's measure on $(-\pi, \pi]$ and the existence of such a density is proven under restrictive summability conditions on the autocovariance operator. With this result, we cannot study processes with seasonal components (whose spectral measure have atoms and therefore no density with respect to Lebesgue's measure) or long-memory processes (for which $\sum_{h\in\mathbb{Z}} \|\Gamma_X(h)\| = +\infty$). In [40], **R1** is proved without the summability assumption but the measure ν_X is constructed via compactification of $\mathcal{L}_b^+(\mathcal{H}_0)$. This compactification makes it possible to define "infinite" operator measures which is not necessary here because p.o.v.m.'s theory is sufficient and makes the construction easier as discussed in Section 6.1.

About **R2** : Assuming ν_X has a density f_X with respect to Lebesgue's measure on $(-\pi, \pi]$, such that $f_X \in L^p((-\pi, \pi], \mathcal{B}((-\pi, \pi]), \mathcal{S}_1(\mathcal{H}_0))$ for some $p \in (1, +\infty]$, [38, Theorem 2.4.3] provides the Stieltjes integral representation for all $t \in \mathbb{Z}$,

$$X_t = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}Z_\lambda \quad \mathbb{P}\text{-a.e.}, \tag{6.4}$$

where $\lambda \mapsto Z_\lambda$ has orthogonal increments. This result is provided without assuming existence of a density with respect to Lebesgue's measure in [40] and is equivalent to **R2** with $\hat{X}((-\pi, \lambda]) = Z_\lambda$ which becomes a c.a.o.s. measure. In [38, Theorem 2.5.1], the author constructs a space (denoted by $\mathfrak{H}$) similar to $\mathsf{L}^2((-\pi,\pi], \mathcal{B}((-\pi,\pi]), \mathcal{O}(\mathcal{H}_0), \nu_X)$ and proves the isometric property of the spectral representation. The difference with the results we present in Sections 2 and Section 3 is that, by making the module structure of $\mathcal{M}(\Omega, \mathcal{F}, \mathcal{H}_0, \mathbb{P})$ explicit, we believe that the construction is clearer and that the objects constructed can be fully characterized. For example, we state in Corollary 3.3 that the spectral representation $\hat{X}$ is a c.a.g.o.s. measure and not only a c.a.o.s. measure and that it defines a gramian-isometry. Moreover, the space $\mathfrak{H}$ of [38] is constructed as the completion of a pre-Hilbert space, that is a quotient space of Cauchy sequences, which provides little intuition on the space of transfer functions one can consider for filtering. On the contrary, the space $\mathsf{L}^2((-\pi,\pi], \mathcal{B}((-\pi,\pi]), \mathcal{O}(\mathcal{H}_0), \nu_X)$ is a space of operator-valued functions which is easier to imagine.

6.4 On the completeness of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$

In Section 2.5, we have defined the normal pre-Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ of square-integrable bounded-operator-valued functions. In the univariate case, this corresponds to $L^2(\mathsf{X}, \mathcal{X}, \nu_X)$ which is a Hilbert space. In the multivariate case, where $\mathcal{H}_0$ and $\mathcal{G}_0$ have finite dimensions, the completeness of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is proven in [33]. However completeness is not guaranteed in the infinite dimensional case, see [27], where the authors refer to [25] for a counter-example. We complete this line of thoughts by providing a necessary and sufficient condition for the completeness of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ in the general case.

Theorem 6.4. *Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces, $(\mathsf{X}, \mathcal{X})$ a measurable space, ν a trace-class p.o.v.m. on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ and $f = \frac{\mathrm{d}\nu}{\mathrm{d}\|\nu\|_1}$. Then $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is complete for the gramian defined in Proposition 2.8 if and only if $\mathrm{rank}(f)$ is finite $\|\nu\|_1$-a.e. In this case, $\mathrm{rank}\left(\frac{\mathrm{d}\nu}{\mathrm{d}\mu}\right)$ is finite μ-a.e. for all finite non-negative measure μ which dominates $\|\nu\|_1$.*

Proof. The proof of the fact that we can take μ instead of $\|\nu\|_1$ uses the same arguments we used to prove Relation (2.12) and will be omitted. Now, let us consider that f is a representing function of the density which is in $\mathcal{S}_1^+(\mathcal{H}_0)$ everywhere and let $A := \{\mathrm{rank} f < +\infty\}$ which is in $\mathcal{X}$ by measurability of the rank (see Proposition A.1) and of f. Then by [18, Theorem 3.1.3], we have $A = \{\mathrm{Im} f^{1/2} \text{ is closed}\}$. We show successively that $\|\nu\|_1(A^c) = 0$ is a necessary condition for completeness of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and then that it is sufficient.
Proof of necessity. Suppose that $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ is complete and that $\|\nu\|_1(A^c) \neq 0$. Then in order to get a contradiction, we will follow the following two steps.

Step 1 Construct a function $\Psi \in L^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1) \subset \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ such that

$$\text{for all } t \in A^c, \quad \Psi(t) \notin \left\{\Phi f(t)^{1/2} \;:\; \Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)\right\} . \tag{6.5}$$

Step 2 Construct a sequence $(\Phi_n)_{n\in\mathbb{N}} \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)^{\mathbb{N}}$ such that $\Phi_n f^{1/2}$ converges to Ψ in $L^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$.

Let us explain why these two steps lead to a contradiction. **Step 2** implies that $(\Phi_n f^{1/2})_{n\in\mathbb{N}}$ is Cauchy in $L^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ which, by the gramian-isometric property shown in Proposition 2.8, means that $(\Phi_n)_{n\in\mathbb{N}}$ is Cauchy in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Since we assumed completeness, there exists $\Phi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ such that Φ_n converges to Φ in this space, which, again by Proposition 2.8, means that $\Phi_n f^{1/2}$ converges to $\Phi f^{1/2}$ in $L^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ and thus $\Phi f^{1/2} = \Psi$ $\|\nu\|_1$-a.e. contradicting (6.5).

We now provide the constructions previously described.
Step 1 By Proposition A.1 and composition of measurable functions, we know that the functions $t \mapsto \lambda_i(t)$ are measurable where $\lambda_i(t)$ is the i-th eigenvalue of $f(t)^{1/2}$ (in decreasing order with the convention of Appendix A.1). Moreover, Proposition A.1 (and again composition of measurable functions) also gives that there exists a family of measurable functions $(\psi_i)_{i\in\mathbb{N}}$

from X to $\mathcal{H}_0$ such that for all $t \in \mathsf{X}$, $(\psi_i(t))_{i\in\mathbb{N}}$ is an orthonormal sequence of eigenvectors associated to the eigenvalues $(\lambda_i)_{i\in\mathbb{N}}$. Define

$$y : t \mapsto \sum_{n\in\mathbb{N}} \ell_n(t)\psi_n(t)$$

with $\ell_n(t) = C(t)^{-1}\lambda_n(t)$ where $C(t) = \left(\sum_{n\in\mathbb{N}} \lambda_n(t)^2\right)^{1/2}$ so that $\|y(t)\|_{\mathcal{H}_0} = 1$. And let

$$\Psi : t \mapsto u \otimes y(t)$$

where $u \in \mathcal{G}_0$ with $\|u\|_{\mathcal{G}_0} = 1$. Then $\Psi \in \mathcal{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ because for all $t \in \mathsf{X}$, $\|\Psi(t)\|_2 = 1$.

We conclude by reasoning by contradiction. Suppose that (6.5) does not hold and take $t \in A^c$ and $\Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0)$ such that $\Psi(t) = \Phi f(t)^{1/2}$. Then we have that $y(t) \otimes u = \Psi(t)^{\mathsf{H}} = f(t)^{1/2}\Phi^{\mathsf{H}}$ and thus

$$y(t) = (y(t) \otimes u)(u) = f(t)^{1/2}\Phi^{\mathsf{H}}u \in \mathrm{Im}(f(t)^{1/2}) .$$

This means that there exists $x \in \mathcal{H}_0$ such that $y(t) = f(t)^{1/2}x$ and we get for all $n \in \mathbb{N}$,

$$C(t)^{-1}\lambda_n(t) = \ell_n(t) = \left\langle f(t)^{1/2}x, \psi_n(t)\right\rangle_{\mathcal{H}_0} = \left\langle x, f(t)^{1/2}\psi_n(t)\right\rangle_{\mathcal{H}_0} = \lambda_n(t)\left\langle x, \psi_n(t)\right\rangle_{\mathcal{H}_0} .$$

In particular $\lambda_n(t) > 0$ implies $\langle x, \psi_n(t)\rangle_{\mathcal{H}_0} = C(t)^{-1}$. Since $\mathrm{rank} f(t) = +\infty$, we know that $\lambda_n(t) > 0$ for all $n \in \mathbb{N}$ and thus get that $\|x\|_{\mathcal{H}_0} = +\infty$, which is impossible.

Step 2 Define

$$\Phi_n : t \mapsto C(t)^{-1}u \otimes \sum_{k=0}^{n} \psi_k(t) .$$

Then $\Phi_n \in \mathcal{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ and $\Phi_n(t)f^{1/2}(t) = u \otimes \sum_{k=0}^{n} \ell_k(t)\psi_k(t)$. Then for all $t \in \mathsf{X}$,

$$\left\|\Psi(t) - \Phi_n(t)f^{1/2}(t)\right\|_2^2 = \sum_{k=n+1}^{+\infty} \ell_k(t)^2 ,$$

which tends to 0 as $n \to +\infty$ and is bounded by 1. Hence by Lebesgue's dominated converge theorem

$$\int \left\|\Psi - \Phi_n f^{1/2}\right\|_2^2 \mathrm{d}\|\nu\|_1 \xrightarrow[n\to+\infty]{} 0 .$$

Proof of sufficiency. Suppose that $\|\nu\|_1(A^c) = 0$, i.e. that $\mathrm{Im} f^{1/2}$ is closed $\|\nu\|_1$-a.e. and consider that $f^{1/2}$ is a representing function of the density which has closed range everywhere. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ and define for all $n \in \mathbb{N}$, $\Psi_n = \Phi_n f^{1/2}$. Then, by Proposition 2.8, $(\Psi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$ which is complete, hence $\Psi = \lim_{n\to+\infty} \Psi_n$ exists in $L^2(\mathsf{X}, \mathcal{X}, \mathcal{S}_2(\mathcal{H}_0, \mathcal{G}_0), \|\nu\|_1)$. This implies that there exists a subsequence $(\Psi_{\phi(n)})_{n\in\mathbb{N}}$ of $(\Psi_n)_{n\in\mathbb{N}}$ which converges $\|\nu\|_1$-a.e. to Ψ. More explicitly, there exists $B \in \mathcal{X}$, with $\|\nu\|_1(B^c) = 0$ such that $\Psi_{\phi(n)}(t) \xrightarrow[n\to+\infty]{} \Psi(t)$ for all $t \in B$. Let $t \in B$, then for all $x \in \mathcal{G}_0$,

$$\Psi(t)^{\mathsf{H}}x = \lim_{n\to+\infty} f(t)^{1/2}\Phi_n^{\mathsf{H}}x \in \overline{\mathrm{Im}} f(t)^{1/2} = \mathrm{Im} f(t)^{1/2} .$$

Hence $\mathrm{Im}\Psi(t)^{\mathsf{H}} \subset \mathrm{Im} f(t)^{1/2} \subset \mathcal{D}\left(f(t)^{1/2}\right)^{-}$ where $\left(f(t)^{1/2}\right)^{-}$ is the generalized inverse of $f(t)^{1/2}$ (see Appendix A.3). This means that we can define

$$\Theta(t) := \left(f(t)^{1/2}\right)^{-} \Psi(t)^{\mathsf{H}} \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{H}_0) .$$

Defining $\Theta(t) = 0$ for $t \in B^c$, we get that $\Theta \in \mathbb{F}_s(\mathsf{X}, \mathcal{X}, \mathcal{G}_0, \mathcal{H}_0)$. This implies that the function

$\Phi : t \mapsto \Theta(t)^{\mathsf{H}}$ is in $\mathbb{F}_s(\mathsf{X}, \mathcal{X}, \mathcal{H}_0, \mathcal{G}_0)$ and we have

$$\begin{aligned}
\int \left\| \Phi(t) f(t)^{1/2} \right\|_2^2 \|\nu\|_1(\mathrm{d}t) &= \int \left\| f(t)^{1/2} \Theta(t) \right\|_2^2 \|\nu\|_1(\mathrm{d}t) \\
&= \int_A \left\| f(t)^{1/2} \left(f(t)^{1/2} \right)^{-} \Psi(t) \right\|_2^2 \|\nu\|_1(\mathrm{d}t) \\
&= \int_A \left\| P_{\mathrm{Im} f(t)^{1/2}} \Psi(t) \right\|_2^2 \|\nu\|_1(\mathrm{d}t) \\
&= \int_A \|\Psi(t)\|_2^2 \|\nu\|_1(\mathrm{d}t) \\
&< +\infty \, .
\end{aligned}$$

Hence $\Phi \in \mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$. Finally

$$\begin{aligned}
\|\Phi - \Phi_n\|^2_{\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)} &= \int \left\| \Phi f^{1/2} - \Phi_n f^{1/2} \right\|_2^2 \mathrm{d}\|\nu\|_1 \\
&= \int \left\| f^{1/2} \Theta - f^{1/2} \Phi_n^{\mathsf{H}} \right\|_2^2 \mathrm{d}\|\nu\|_1 \\
&= \int_A \left\| \Psi^{\mathsf{H}} - \Psi_n^{\mathsf{H}} \right\|_2^2 \mathrm{d}\|\nu\|_1 \\
&= \int_A \|\Psi - \Psi_n\|_2^2 \mathrm{d}\|\nu\|_1 \\
&\xrightarrow[n \to +\infty]{} 0
\end{aligned}$$

that is, $(\Phi_n)_{n \in \mathbb{N}}$ converges to Φ in $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$ thus concluding the proof of completeness of $\mathsf{L}^2(\mathsf{X}, \mathcal{X}, \mathcal{L}_b(\mathcal{H}_0, \mathcal{G}_0), \nu)$. $\square$

A Useful functional analysis results

A.1 Diagonalization of compact positive operators and measurability results

Let $\mathcal{H}_0$ be a separable Hilbert space and $\Phi \in \mathcal{L}_b(\mathcal{H}_0)$. Then $\lambda \in \mathbb{C}$ is said to be an *eigenvalue* of Φ if $\ker(\Phi - \lambda \mathrm{Id}) \neq \{0\}$. If λ is an eigenvalue of Φ, we say that $\ker(\Phi - \lambda \mathrm{Id})$ is the associated *eigensubspace* and its dimension is called the *multiplicity* of λ. We denote by $\mathrm{spec_p}(\Phi)$ the set of eigenvalues of Φ (called the *point spectrum* of Φ). Φ is said to be *diagonalizable* if $\mathcal{H}_0$ has a Hilbert-basis of eigenvectors of Φ. If $\Phi \in \mathcal{K}(\mathcal{H}_0)$ and is auto-adjoint, then it is diagonalizable and $\mathrm{spec_p}(\Phi)$ is at most discrete, every non-zero eigenvalue has finite dimension and eigensubspace associated to different eigenvalues are orthogonal. We denote by $N_{\mathrm{sp}}(\Phi)$ the cardinal of $\mathrm{spec_p}(\Phi)$ which is finite if and only if $\mathrm{rank}(\Phi) < +\infty$ and if not, then $\mathrm{spec_p}(\Phi)$ admits 0 as its unique accumulation point (equivalently, this means that any way of representing the elements of $\mathrm{spec_p}(\Phi)$ gives a sequence converging to 0).

In order to have a representation which takes into account both cases we add zeros at the end of the sequence in the case where $N_{\mathrm{sp}}(\Phi) < +\infty$. This way, we can always represent the eigenvalues of Φ as a sequence converging to 0. When $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$ all its eigenvalues are non-negative and it is convenient to represent them in decreasing order which, in the case where $N_{\mathrm{sp}}(\Phi) = +\infty$, gives a sequence of strictly positive numbers decreasing to 0 even if $0 \in \mathrm{spec_p}(\Phi)$. We will denote by $(\lambda_i(\Phi))_{i \in \mathbb{N}}$ such a sequence of distinct eigenvalues, that is if $N_{\mathrm{sp}}(\Phi) < +\infty$, then $\lambda_0(\Phi) > \lambda_1(\Phi) > \cdots > \lambda_{N_{\mathrm{sp}}(\Phi)}(\Phi) > 0$ and $\lambda_i(\Phi) = 0$ for all $i > N_{\mathrm{sp}}(\Phi)$ and if $N_{\mathrm{sp}}(\Phi) = +\infty$, then $\lambda_0(\Phi) > \lambda_1(\Phi) > \cdots > 0$. In the latter case, one need to keep in mind the fact that 0 can be an eigenvalue even if it is not represented in the sequence. Using this representation $(\lambda_i(\Phi))_{i \in \mathbb{N}}$, we will also denote by $m_i(\Phi)$ the multiplicity of $\lambda_i(\Phi)$ and by $\Pi_i(\Phi)$ the orthogonal projection onto $\ker(\Phi - \lambda_i(\Phi)\mathrm{Id})$ for all $i \in \mathbb{N}$. Finally we define $(\alpha_i(\Phi))_{i \in \mathbb{N}}$ the piecewise constant sequence obtained by repeating the values of $(\lambda_i(\Phi))_{i \in \mathbb{N}}$ as often as their multiplicities. With these notations we can write

$$\Phi = \sum_{i \in \mathbb{N}} \lambda_i(\Phi) \Pi_i(\Phi) \tag{A.1}$$

where the series converges in operator norm, and if $\Phi \neq 0$,

$$\mathrm{Id} = P_{\ker(\Phi)} + P_{\overline{\mathrm{Im}(\Phi)}} = P_{\ker(\Phi)} + \sum_{0 \leq i < N_{\mathrm{sp}}(\Phi)} \Pi_i \tag{A.2}$$

where, if $N_{\mathrm{sp}}(\Phi) = +\infty$, the series converges in s.o.t. (If $\Phi = 0$ we have $\mathrm{Id} = P_{\ker(\Phi)} = \Pi_i$ for all $i \in \mathbb{N}$). Moreover the following measurability properties hold (recall that the notion of simple measurability is defined in Section 2.1 and $\mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)) = \{A \cap \mathcal{K}^+(\mathcal{H}_0) \;:\; A \in \mathcal{B}(\mathcal{K}(\mathcal{H}_0))\}$).

Proposition A.1. *The following assertions hold for all $i \in \mathbb{N}$.*

(i) $\alpha_i : \Phi \mapsto \alpha_i(\Phi)$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$.

(ii) $m_i : \Phi \mapsto m_i(\Phi)$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\overline{\mathbb{N}}, \mathcal{P}(\overline{\mathbb{N}}))$.

(iii) $\lambda_i : \Phi \mapsto \lambda_i(\Phi)$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$.

(iv) $\mathrm{rank} : \Phi \mapsto \mathrm{rank}(\Phi)$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\overline{\mathbb{N}}, \mathcal{P}(\overline{\mathbb{N}}))$.

(v) $N_{\mathrm{sp}} : \Phi \mapsto N_{\mathrm{sp}}(\Phi)$ *is measurable from* $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\overline{\mathbb{N}}, \mathcal{P}(\overline{\mathbb{N}}))$.

(vi) $\Pi_i : \Phi \mapsto \Pi_i(\Phi)$ *is simply measurable from* $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $\mathcal{L}_b(\mathcal{H}_0)$.

(vii) $\Phi \mapsto P_{\ker(\Phi)}$ *is simply measurable from* $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $\mathcal{L}_b(\mathcal{H}_0)$.

(viii) There exists a family $(\psi_i)_{i \in \mathbb{N}}$ *of functions* $\psi_i : \Phi \mapsto \psi_i(\Phi)$ *which are measurable from* $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ *to* $(\mathcal{H}_0, \mathcal{B}(\mathcal{H}_0))$ *such that* $\forall \Phi \in \mathcal{K}^+(\mathcal{H}_0)$, $(\psi_i(\Phi))_{i \in \mathbb{N}}$ *is orthonormal and* $\forall i \in \mathbb{N}$, $\psi_i(\Phi) \in \ker(\Phi - \lambda_i(\Phi)\mathrm{Id})$.

Proof. We follow the ideas of the proofs of [27, Theorem 2.10] and [20, Lemma 3.4.7].
Proof of (i). By [20, Lemma 3.4.6], for all $n \in \mathbb{N}$ and all $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$,

$$\sum_{i=0}^{n} \alpha_i(\Phi) = \max \left\{ \sum_{i=0}^{n} \langle \Phi x_i, x_i \rangle_{\mathcal{H}_0} \;:\; (x_0, \cdots, x_n) \text{ is orthonormal} \right\}$$

and therefore $\sum_{i=0}^{n} \alpha_i$ is measurable from $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then using $\alpha_i = \sum_{j=1}^{i} \alpha_j - \sum_{j=1}^{i-1} \alpha_j$ we get measurability of α_i for all $i \in \mathbb{N}$.
Proof of (ii). By definition, for all $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$, $m_0(\Phi) = \inf \{j \in \mathbb{N} \;:\; \alpha_j(\Phi) \neq \alpha_{j+1}(\Phi)\}$ with the convention $\inf \emptyset = +\infty$ and for all $i \in \mathbb{N}^*$,

$$m_i(\Phi) = \begin{cases} \inf \{j > m_{i-1}(\Phi) \;:\; \alpha_j(\Phi) \neq \alpha_{j+1}(\Phi)\} - m_{i-1}(\Phi) & \text{if } m_{i-1}(\Phi) < +\infty \\ +\infty & \text{otherwise} \end{cases}.$$

Measurability of the m_i's is then proven by induction.
Proof of (iii). For all $i \in \mathbb{N}$, for all $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$, $\lambda_i(\Phi) = \alpha_{m_i(\Phi)}(\Phi) \mathbb{1}_{\{m_i(\Phi) \neq 0\}}$ hence λ_i is measurable.
Proof of (iv). For all $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$, $\mathrm{rank}(\Phi) = \sum_{i \in \mathbb{N}} \mathbb{1}_{\{\alpha_i(\Phi) \neq 0\}}$ hence rank is measurable.
Proof of (v). For all $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$, $N_{\mathrm{sp}}(\Phi) = \sum_{i \in \mathbb{N}} \mathbb{1}_{\{\lambda_i(\Phi) \neq 0\}}$ hence N_{sp} is measurable.
Proof of (vi). Let $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$, then from (A.1) one can show that for all $n \in \mathbb{N}$,

$$\left(\frac{\Phi}{\lambda_0(\Phi)} \right)^n = \sum_{k \in \mathbb{N}} \left(\frac{\lambda_k(\Phi)}{\lambda_0(\Phi)} \right)^n \Pi_k(\Phi) \quad \text{in s.o.t.}$$

and for all $1 \leq i < N_{\mathrm{sp}}(\Phi)$,

$$\left(\frac{\Phi - \sum_{k=0}^{i-1} \lambda_k \Pi_k}{\lambda_i} \right)^n = \sum_{k \in \mathbb{N}} \left(\frac{\lambda_k(\Phi)}{\lambda_i(\Phi)} \right)^n \Pi_k \quad \text{in s.o.t.}$$

From these two relations and (A.2) we easily get

$$\Pi_0(\Phi) = \lim_{n \to +\infty} \left(\frac{\Phi}{\lambda_0(\Phi)} \right)^n \mathbb{1}_{\{\lambda_0(\Phi) \neq 0\}} + \mathrm{Id} \mathbb{1}_{\{\lambda_0(\Phi) = 0\}}$$

and for all $i \geq 1$,

$$\begin{aligned}\Pi_i(\Phi) = &\operatorname{Id}\mathbb{1}_{\{\lambda_0(\Phi)=0\}}\\ &+ \mathbb{1}_{\{\lambda_0(\Phi)\neq 0\}}\mathbb{1}_{\{i<N_{\mathrm{sp}}(\Phi)\}}\lim_{n\to+\infty}\left(\frac{\Phi - \sum_{k=0}^{i-1}\lambda_k(\Phi)\Pi_k(\Phi)}{\lambda_i(\Phi)}\right)^n\\ &+ \mathbb{1}_{\{\lambda_0(\Phi)\neq 0\}}\mathbb{1}_{\{i\geq N_{\mathrm{sp}}(\Phi)\}}\left(\operatorname{Id} - \sum_{0\leq k<N_{\mathrm{sp}}(\Phi)}\Pi_k(\Phi)\right)\end{aligned}$$

where the convergences are in s.o.t. Hence by measurability of the λ_i's and of N_{sp} we get by induction than the Π_i's are simply measurable as limit in s.o.t. of simply measurable functions.

Proof of (vii). Simple measurability of $\Phi \mapsto P_{\ker(\Phi)}$ comes from (A.2), simple measurability of the Π_i's and measurability of N_{sp}.

Proof of (viii). Let $(\phi_n)_{n\in\mathbb{N}}$ be a Hilbert-basis of $\mathcal{H}_0$, then define for all

$$\tau_i : \Phi \mapsto \begin{cases}\min\{n\in\mathbb{N} \,:\, \Pi_i(\Phi)\phi_n \neq 0\} & 0\leq i<N_{\mathrm{sp}}(\Phi)\\ \min\left\{n>\tau_{i-1}(\Phi) \,:\, P_{\ker(\Phi)}\phi_n\neq 0\right\} & i\geq N_{\mathrm{sp}}(\Phi)\end{cases}.$$

Note that τ_i never takes the value $+\infty$ because for all $i\in\mathbb{N}$, $\Pi_i(\Phi)\neq 0$ and if $N_{\mathrm{sp}}(\Phi) < +\infty$, then $\ker(\Phi)$ has infinite dimension and therefore there are infinitely many $n\in\mathbb{N}$ such that $P_{\ker(\Phi)}\phi_n \neq 0$. Then measurability of N_{sp} and simple measurability of the Π_i's give that the τ_i's are measurable from $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0)))$ to $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$. Now define for all $i\in\mathbb{N}$, $\varphi_i : \Phi\mapsto \Pi_i(\Phi)\phi_{\tau_i(\Phi)}$ and the sequence $(\tilde{\psi}_i)_{i\in\mathbb{N}}$ obtained by applying the Gram-Schmidt algorithm to the φ_i's, that is $\tilde{\psi}_0 : \Phi\mapsto\varphi_0(\Phi)$ and for all $i\geq 1$, $\tilde{\psi}_i : \Phi \mapsto \varphi_i(\Phi) - \sum_{k=0}^{i-1}\frac{\langle\varphi_i(\Phi),\tilde{\psi}_k(\Phi)\rangle}{\|\tilde{\psi}_k(\Phi)\|^2_{\mathcal{H}_0}}\tilde{\psi}_k(\Phi)$. Finally, define for all $i\in\mathbb{N}$, $\psi_i : \Phi\mapsto \tilde{\psi}_i(\Phi)/\left\|\tilde{\psi}_i(\Phi)\right\|_{\mathcal{H}_0}$. Then, measurability of the φ_i's implies measurability of the ψ_i's and, by construction for all $\Phi\in\mathcal{K}^+(\mathcal{H}_0)$, the family $(\psi_i(\Phi))_{i\in\mathbb{N}}$ is orthonormal.

□

A.2 Singular values decomposition

Let $\mathcal{H}_0$, $\mathcal{G}_0$ be two separable Hilbert spaces and $\Phi\in\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)$, then $\Phi^H\Phi\in\mathcal{K}^+(\mathcal{H}_0)$ and $\Phi\Phi^H\in\mathcal{K}^+(\mathcal{G}_0)$ and these two operators have the same non-zero eigenvalues with the same (finite) multiplicities.

Define the set $\operatorname{sing}(\Phi)$ of *singular values* of $\Phi\in\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)$ as

$$\operatorname{sing}(\Phi) = \left\{\lambda^{1/2} \,:\, \lambda\in\operatorname{spec}_{\mathrm{p}}(\Phi^H\Phi)\setminus\{0\}\right\} = \left\{\lambda^{1/2} \,:\, \lambda\in\operatorname{spec}_{\mathrm{p}}(\Phi\Phi^H)\setminus\{0\}\right\}$$

and for all $\sigma\in\operatorname{sing}(\Phi)$ we call *multiplicity* of σ the multiplicity of σ^2 as an eigenvalue of $\Phi^H\Phi$ or $\Phi^H\Phi$ (which are the same). The well-known singular value decomposition theorem can then be stated as follows.

Theorem A.2 (Singular value decomposition)**.** *Let $\mathcal{H}_0$ and $\mathcal{G}_0$ be two separable Hilbert spaces and $\Phi\in\mathcal{K}(\mathcal{H}_0,\mathcal{G}_0)$ then there exist two Hilbert basis $(\phi_n)_{0\leq n<\operatorname{rank}(\Phi)}$ and $(\psi_n)_{0\leq n<\operatorname{rank}(\Phi)}$ of $\overline{\operatorname{Im}(\Phi^H)}$ and $\overline{\operatorname{Im}(\Phi)}$ respectively and $(\sigma_n)_{0\leq n<\operatorname{rank}(\Phi)}$ representing the elements of* $\operatorname{sing}(\Phi)$ *repeated as often as their multiplicity such that*

$$\Phi = \sum_{0\leq n<\operatorname{rank}(\Phi)}\sigma_n\psi_n\otimes\phi_n \tag{A.3}$$

where the series converges in operator norm. Moreover, $\lim_{n\to+\infty}\sigma_n = 0$.

Similarly to the eigendecomposition, the singular values are usually written as a decreasing sequence $(\sigma_i(\Phi))_{0\leq i<\operatorname{rank}(\Phi)}$.

A.3 Generalized inverse of an operator

Let $\mathcal{H}_0, \mathcal{G}_0$ be separable Hilbert spaces and $\Phi \in \mathcal{L}_b(\mathcal{H}_0, \mathcal{H}_0)$, then the mapping

$$\Phi_{|\ker(\Phi)^\perp \to \mathrm{Im}(\Phi)} : \begin{array}{ccc} \ker(\Phi)^\perp & \to & \mathrm{Im}(\Phi) \\ x & \mapsto & \Phi x \end{array}$$

is an isomorphism and we define $\Phi^- \in \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0)$ (called the *generalized inverse* of Φ) as the linear extension of $\left(\Phi_{|\ker(\Phi)^\perp \to \mathrm{Im}(\Phi)}\right)^{-1}$ to $\mathcal{D}(\Phi^-) := \mathrm{Im}(\Phi) \oplus \mathrm{Im}(\Phi)^\perp$ such that $\ker(\Phi^-) = \mathrm{Im}(\Phi)^\perp$. In other words, for all $x \in \mathcal{D}(\Phi^-)$, there exists $(x_1, x_2) \in \mathrm{Im}(\Phi) \times \mathrm{Im}(\Phi)^\perp$ such that $x = x_1 + x_2$, then $\Phi^- x = \left(\Phi_{|\ker(\Phi)^\perp \to \mathrm{Im}(\Phi)}\right)^{-1} x_1$.

The subspace $\mathcal{D}(\Phi)$ is dense in $\mathcal{G}_0$ and is equal to $\mathcal{G}_0$ if and only if $\mathrm{Im}(\Phi)$ is closed, in which case $\Phi^- \in \mathcal{L}_b(\mathcal{G}_0, \mathcal{H}_0)$. The operators Φ and Φ^- are linked by the relation

$$\Phi^- \Phi = P_{\ker(\Phi)^\perp} \tag{A.4}$$

and it is easy to show that, if $\Psi \in \mathcal{O}(\mathcal{G}_0, \mathcal{H}_0)$, then $\Psi = \Phi^-$ if and only if $\Psi\Phi = P_{\ker(\Phi)^\perp}$ and $\Psi_{|\ker(\Phi)^\perp} = 0$.

The identity (A.4), along with the fact that a projection is compact if and only if it has finite rank, gives (see [18, Theorem 3.1.3]) that a compact operator has closed range if and only if it has finite rank. The operator $\Phi\Phi^-$ is not as easy to characterize but when $\mathrm{Im}(\Phi)$ is closed, we have $\Phi\Phi^- = P_{\mathrm{Im}(\Phi)}$. Finally, in the case where $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$, the generalized inverse can be diagonalized as follows.

Proposition A.3. *Let $\mathcal{H}_0$ be a separable Hilbert space and $\Phi \in \mathcal{K}^+(\mathcal{H}_0)$, then, defining for all $i \in \mathbb{N}$, $\lambda_i^-(\Phi) = 1/\lambda_i(\Phi)$ if $\lambda_i(\Phi) \neq 0$ and 0 otherwise, we get*

$$\mathcal{D}(\Phi^-) = \left\{ x \in \mathcal{H}_0 \;:\; \sum_{i \in \mathbb{N}} \left(\lambda_i^-(\Phi)\right)^2 \|\Pi_i(\Phi) x\|_{\mathcal{H}_0}^2 < +\infty \right\} \tag{A.5}$$

and for all $x \in \mathcal{D}(\Phi^-)$,

$$\Phi^- x = \sum_{i \in \mathbb{N}} \lambda_i^-(\Phi) \Pi_i(\Phi) x \tag{A.6}$$

Proof. The inclusion ($\subset$) in (A.5) is straightforward. To show the converse inclusion, let $x \in \mathcal{H}_0$ such that $\sum_{i \in \mathbb{N}} \left(\lambda_i^-(\Phi)\right)^2 \|\Pi_i(\Phi) x\|_{\mathcal{H}_0}^2 < +\infty$, then $y := \sum_{i \in \mathbb{N}} \lambda_i(\Phi)^- \Pi_i(\Phi) x$ exists because the series converges in $\mathcal{H}_0$. Now, we write $x = P_{\overline{\mathrm{Im}(\Phi)}} x + P_{\mathrm{Im}(\Phi)^\perp x}$ with

$$P_{\overline{\mathrm{Im}(\Phi)}} x = \sum_{0 \le i < N_{\mathrm{sp}}(\Phi)} \Pi_i(\Phi) x = \sum_{0 \le i < N_{\mathrm{sp}}(\Phi)} \lambda_i(\Phi) \lambda_i(\Phi)^- \Pi_i(\Phi) x = \Phi y \in \mathrm{Im}(\Phi) \;,$$

and therefore $x \in \mathcal{D}(\Phi^-)$ which concludes the proof of (A.5).

To show (A.6), let $x \in \mathcal{D}(\Phi^-)$ and define the operator

$$\Psi : \begin{array}{ccc} \mathcal{D}(\Phi^-) & \to & \mathcal{H}_0 \\ x & \mapsto & \sum_{i \in \mathbb{N}} \lambda_i^-(\Phi) \Pi_i(\Phi) x \end{array}$$

Then, it is easy to verify that $\Psi\Phi = P_{\overline{\mathrm{Im}(\Phi)}} = P_{\ker(\Phi)^\perp}$ and that $\Psi_{|\mathrm{Im}(\Phi)^\perp} = 0$ which imply that $\Psi = \Phi^-$. ◻

Using this result, we can show the useful measurability property.

Corollary A.4. *Let $\mathcal{H}_0$ be a separable Hilbert space. Then the mapping $\Phi \mapsto \Phi^-$ is $\mathcal{O}$-measurable (see Section 2.1 for the definition) from $(\mathcal{K}^+(\mathcal{H}_0), \mathcal{B}(\mathcal{K}^+(\mathcal{H}_0))$ to $\mathcal{O}(\mathcal{H}_0)$*

Proof. Measurability of the λ_i's and simple measurability of the Π_i's shown in Proposition A.1 give Condition (i) of the definition of $\mathcal{O}$-measurability using (A.5) and Condition (ii) using (A.6). ◻

B Locally compact Abelian groups

A topological group is a group $(\mathrm{T},+)$ (with null element 0) endowed with a topology for which the addition and the inversion maps are continuous in $\mathrm{T}\times\mathrm{T}$ and T respectively. If T is Abelian (i.e. commutative) and is locally compact, Hausdorff for its topology, then it is called a Locally compact Abelian (l.c.a.) group. The dual group $\hat{\mathrm{T}}$ of an l.c.a. group T is the set of continuous characters of T. A character χ of T is a group homomorphism from T to the unit circle group $\mathbb{U} := \{z \in \mathbb{C} : |z| = 1\}$ that is $\chi : \mathrm{T} \to \mathbb{U}$ and for all $s,t \in \mathrm{T}$, $\chi(s+t) = \chi(s)\chi(t)$. In particular, $\chi(0) = 1$ and $\overline{\chi(t)} = \chi(t)^{-1} = \chi(-t)$ for all $t \in \mathrm{T}$. $\hat{\mathrm{T}}$ is a multiplicative Abelian group if we define the product of $\chi_1,\chi_2 \in \hat{\mathrm{T}}$, as $\chi_1\chi_2 : t \mapsto \chi_1(t)\chi_2(t)$, the identity element as $\hat{e} : t \mapsto 1$ and the inverse of $\chi \in \hat{\mathrm{T}}$ as $\chi^{-1} : t \mapsto \chi(t)^{-1} = \overline{\chi(t)}$. $\hat{\mathrm{T}}$ becomes an l.c.a. group when endowed with the compact-open topology, that is the topology for which $\chi_n \to \chi$ in $\hat{\mathrm{T}}$ if and only if for every compact $K \subset \mathrm{T}$, $\chi_n \to \chi$ uniformly on K i.e. $\sup_{t\in K}|\chi_n(t) - \chi(t)| \xrightarrow[n\to+\infty]{} 0$.

A result known as the Pontryagin Duality Theorem (see [34, Theorem 1.7.2]) states that T and $\hat{\hat{\mathrm{T}}}$ are isomorphic via the evaluation map $\begin{array}{ccc} \mathrm{T} & \to & \hat{\hat{\mathrm{T}}} \\ t & \mapsto & \mathrm{e}_t \end{array}$ where $\mathrm{e}_t : \chi \mapsto \chi(t)$ in the sense that this map is a bijective continuous homomorphisms with continuous inverse. In particular, this means that $\{\mathrm{e}_t : t \in \mathrm{T}\}$ is the set of characters of $\hat{\mathrm{T}}$ (i.e. $\hat{\hat{\mathrm{T}}}$). The following theorem will be very useful.

Theorem B.1. *Let* T *be an l.c.a. group and* μ *a regular finite non-negative measure on* $(\mathrm{T},\mathcal{B}(\mathrm{T}))$. *Then for all Banach space* E,

$$L^2(\mathrm{T},\mathcal{B}(\mathrm{T}),E,\mu) = \overline{\mathrm{Span}}^{L^2(\mathrm{T},\mathcal{B}(\mathrm{T}),E,\mu)}\left(t \mapsto \chi(t)x \,:\, \chi \in \hat{\mathrm{T}} x \in E\right)$$

Proof. The space $\mathrm{Span}\left(\hat{\mathrm{T}}\right)$ satisfies the conditions of the Stone-Weierstrass theorem (see [10]) and therefore is uniformly dense in $\mathcal{C}_0(\mathrm{T}) \supset \mathcal{C}_c(\mathrm{T})$. This implies that $\mathrm{Span}\left(t \mapsto \chi(t)x \,:\, \chi \in \hat{\mathrm{T}}, x \in E\right)$ is uniformly dense in $\mathrm{Span}\left(t \mapsto f(t)x \,:\, f \in \mathcal{C}_c(\mathrm{T}), x \in E\right)$ which is itself uniformly dense in $\mathcal{C}_c(\mathrm{T},E)$ by [39, Proposition 44.2]. Since μ is finite, uniform density implies density in L^2-norm and therefore we have shown that $\mathrm{Span}\left(t \mapsto \chi(t)x \,:\, \chi \in \hat{\mathrm{T}}, x \in E\right)$ is dense in $\mathcal{C}_c(\mathrm{T},E)$ in L^2-norm. The result follows because, since μ is regular, $\mathcal{C}_c(\mathrm{T},E)$ is dense in $L^2(\mathrm{T},\mathcal{B}(\mathrm{T}),E,\mu)$ for the L^2-norm. □

It is straightforward to verify that $\mathbb{Z}$ is an l.c.a. group for the addition and discrete topology (the open sets are the subsets of $\mathbb{Z}$, in this case every mapping from $\mathbb{Z}$ to any topological space is continuous). Then $\chi \in \hat{\mathbb{Z}}$ if and only if for all $t,s \in \mathbb{Z}$, $\chi(t+s) = \chi(t)\chi(s)$ and therefore $\hat{\mathbb{Z}} = \left\{ \begin{array}{ccc} \mathbb{Z} & \to & \mathbb{U} \\ t & \mapsto & z^t \end{array} \,:\, z \in \mathbb{U}\right\}$. Since the compact sets of $\mathbb{Z}$ are the finite subsets of $\mathbb{Z}$, the compact-open topology on $\hat{\mathbb{Z}}$ is the same as the one induced by pointwise convergence. It is then easy to show that $\hat{\mathbb{Z}}$, $\mathbb{U}$ and $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ are isomorphic (from $\hat{\mathbb{Z}}$ to $\mathbb{U}$ take $\chi \mapsto \chi(1)$ and from $\mathbb{T}$ to $\mathbb{U}$ take $\lambda \mapsto \mathrm{e}^{\mathrm{i}\lambda}$). In this case we identify $\hat{\mathbb{Z}}$ and $\mathbb{T}$ which is in general represented by $(-\pi,\pi]$. The other classical example of l.c.a. group is $\mathbb{R}$ for the addition and usual topology. It can be shown that $\hat{\mathbb{R}} = \left\{ \begin{array}{ccc} \mathbb{R} & \to & \mathbb{U} \\ t & \mapsto & \mathrm{e}^{\mathrm{i}t\lambda} \end{array} \,:\, \lambda \in \mathbb{R}\right\}$ (see for example [9, Theorem 9.11.] where the idea is to show that the fact that $\chi \in \hat{\mathbb{R}}$ satisfies $\chi(t+s) = \chi(t)\chi(s)$ implies that χ must be differentiable and satisfies a first order differential equation leading to the result). Then $\hat{\mathbb{R}}$ and $\mathbb{R}$ are isomorphic via the mapping $\lambda \mapsto (t \mapsto \mathrm{e}^{\mathrm{i}t\lambda})$. In this case we identify $\hat{\mathbb{R}}$ and $\mathbb{R}$.

C Some particular operator and vector valued measures

C.1 Projection-valued and gramian-projection-valued measures

Let $(\mathsf{X}, \mathcal{X})$ be a measurable space and $\mathcal{H}_0$ a separable Hilbert space. A *projection-valued measure* (p.v.m.) ξ on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ is a p.o.v.m. valued in the space of orthogonal projections on $\mathcal{H}_0$. If in addition $\xi(\mathsf{X}) = \mathrm{Id}$, we say that ξ is *normalized*. This notion appears in diagonalization of non-compact operators and in Stone's theorem where such measures are often mentioned as "spectral measures" or "spectral operator measures" (see e.g. [9, Chapter IX]) but it must not be mistaken with what we defined as "spectral operator measures" for weakly stationary stochastic process. When working with modules, the notion of p.v.m.'s can be extended to *gramian-projection-valued measures* (g.p.v.m.) which play the same role as p.v.m.'s for the extension of Stone's theorem on modules. If $\mathcal{H}$ is a normal Hilbert $\mathcal{L}_b(\mathcal{H}_0)$-module, then a p.v.m. ξ on $(\mathsf{X}, \mathcal{X}, \mathcal{H})$ is said to be a g.p.v.m. if for all $A \in \mathcal{X}$, $\xi(A)$ is a gramian-projection. The notion of regularity used for p.v.m.'s and g.p.v.m.'s is the one defined in Section 2.2 for p.o.v.m.'s.

C.2 Countably additive orthogonally scattered measures

Let $\mathcal{H}_0$ be a Hilbert space and $(\mathsf{X}, \mathcal{X})$ a measurable space. A countably additive orthogonally scattered (c.a.o.s.) measure W on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ is an $\mathcal{H}_0$-valued measure which satisfies for all $A, B \in \mathcal{X}$ such that $A \cap B = \emptyset$, $\langle W(A), W(B)\rangle_{\mathcal{H}_0} = 0$. The proofs of the following assertions are straightforward.

(i) If W is a c.a.o.s. measure on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$, then $\eta_W : A \mapsto \|W(A)\|^2_{\mathcal{H}_0}$ is a finite, non-negative measure on $(\mathsf{X}, \mathcal{X})$ called the *intensity measure* of W. It satisfies for all $A, B \in \mathcal{X}$,
$$\eta_W(A \cap B) = \langle W(A), W(B)\rangle_{\mathcal{H}_0} \ .$$

(ii) Conversely, if $W : \mathcal{X} \to \mathcal{H}_0$ is such that there exists a finite, non-negative measure η on $(\mathsf{X}, \mathcal{X})$ satisfying $\forall A, B \in \mathcal{X}$, $\langle W(A), W(B)\rangle_{\mathcal{H}_0} = \eta(A \cap B)$. Then W is a c.a.o.s. measure on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ with intensity measure η.

When X is a locally-compact topological space then, by definition of the intensity measure, we get that a c.a.o.s. measure W is regular (in the sense recalled in Section 2) if and only if its intensity measure is regular. Since we do not assume that a c.a.o.s. measure has finite variation, we cannot use Bochner's integration theory recalled in Section 2.1. However, Assertion (i) implies that we can linearly, continuously and isometrically extend the mapping $\mathbb{1}_A \mapsto W(A)$ to
$$\overline{\mathrm{Span}}\left(1_A\ ,\ A \in \mathcal{X}\right) = L^2\left(\mathsf{X}, \mathcal{X}, \eta_W\right) \ .$$
That is, there exists a unique isometric operator $I_W : L^2(\mathsf{X}, \mathcal{X}, \eta_W) \to \mathcal{H}_0$ such that $\forall A \in \mathcal{X}$, $I_W(\mathbb{1}_A) = W(A)$. Moreover, I_W is unitary from $L^2(\mathsf{X}, \mathcal{X}, \eta_W)$ to $\overline{\mathrm{Span}}^{\mathcal{H}_0}\left(W(A), A \in \mathcal{X}\right)$ and we define integration of $L^2(\mathsf{X}, \mathcal{X}, \eta_W)$ functions with respect to W by setting, for all $f \in L^2(\mathsf{X}, \mathcal{X}, \eta_W)$,
$$\int f \mathrm{d}W := I_W(f) \ .$$
Conversely, if η is a finite, non-negative measure on $(\mathsf{X}, \mathcal{X})$ and I is an isometry from $L^2(\mathsf{X}, \mathcal{X}, \eta)$ to $\mathcal{H}_0$, then there exists a unique c.a.o.s. measure W on $(\mathsf{X}, \mathcal{X}, \mathcal{H}_0)$ with intensity measure η such that, for all $f \in L^2(\mathsf{X}, \mathcal{X}, \eta)$,
$$w(f) = \int f \, \mathrm{d}W \ .$$

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